"EVERYTHING" IS A BOUNDARY CONDITION OF AN AKSZ MODEL

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JOINT WORK NOT YET IN PROGRESS WITH JÁN PULMANN AND FRIDRICH VALACH

Two motivating examples (2d models on a boundary)

W - 3d oriented, $\Sigma = \partial W$ with a cpx structure

Chern-Simons & chiral WZW

$$S(A) = \int_W \frac{1}{2}\langle A, dA\rangle + \frac{1}{6}\langle A, [A,A]\rangle$$

$$A \in \Omega(W, \mathfrak{g}[1])$$

$$A|_\Sigma \in \Omega^{1,0}(\Sigma, \mathfrak{g}[1]) \oplus \Omega^2(\Sigma, \mathfrak{g}[1])$$

more general: $\Omega^{1,0}(\Sigma, C_+[1]) \oplus \Omega^{0,1}(\Sigma, C_-[1])$

$$\mathfrak{g} = C_+ \oplus C_-, \quad C_- = C_+^{\perp}$$

no boundary term in $S(A)$

2d σ-models (e.g. WZW)

$M, g, H \in \Omega^3_{cl}(M)$

$$\begin{tikzcd} \Sigma \arrow[r, "f"] \arrow[d, phantom, "\cap" rotate=90] & M \\ W \arrow[ur, "f"'] & \end{tikzcd}$$

$$S(f) = \int_\Sigma g(\partial f, \bar\partial f) + \int_W f^* H$$

$S(f)$ is mostly the boundary term

but Courant σ-model will cure that

What we get in general? (many promises)

dim $W = n+1$, $\Sigma = \partial W$ (+ a geometric structure on Σ, e.g. Riem. metric)

(X, ω_X, Q_X) a dg symplectic manifold, deg $\omega_X = n$

a suitable boundary condition on Σ ($\mathcal{L} \subset Maps(T[1]\Sigma, X)$ dg Lagrangian)

$n=1$ bulk: Poisson σ-model ($X = T^*[1]M$, π Poisson structure on M)

b.c. $\to$ Hamiltonian mechanics on M (or reduction)

standalone if M is symplectic

$n=2$ bulk: Courant σ-model, e.g. Chern-Simons

b.c.: chiral/antichiral (conformal structure on Σ), potential/2d YM (Riemannian str.)

= gauge & σ-model-type theories standalone if the CA is exact

$n \geq 3$ all kind of σ-models/higher gauge theories (using differential forms)

All this in a BV setup, with a propagator (ready for perturbative calculations)

"Good" boundary conditions (fewer promises)

= b.c.'s involving no derivatives, giving elliptic complexes (⇝ propagator)

a silly simple case (but almost the full story): everything linear

$\Sigma = V$ a n-dim vector space, (X, ω_X, Q_X) a complex with a pairing

⇝ space of boundary fields $\Omega(V) \otimes X = C^\infty(V) \otimes \Lambda V^* \otimes X$

boundary condition = a graded vect. subspace $\Lambda \subset \Lambda V^* \otimes X$

the b.c. is $\mathcal{L} = C^\infty(V) \otimes \Lambda \subset \Omega(V) \otimes X$

observation:

1) $\mathcal{L} \subset \Omega(V) \otimes X$ is a Lagrangian subcomplex iff
$\Lambda \subset \Lambda V^* \otimes X$ is a Lagrangian subcomplex and a ΛV^*-submodule

2) $\mathcal{L}$ is an elliptic complex iff $(\Lambda, \alpha\wedge)$ is acyclic $\forall\, 0 \neq \alpha \in V^*$
(propagator: contracting homotopy for $(\Omega(W) \otimes X)^{b.c.}$)

Boundary conditions and G-structures on Σ

A non-linear version needs:

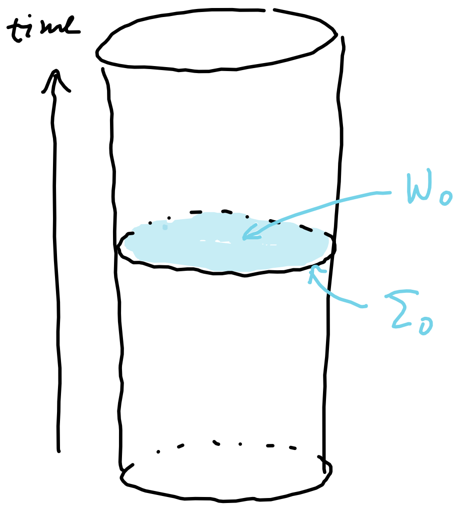
- $G \subset GL(V)$
- a torsion-free G-structure on Σ (e.g. Riemannian metric)
- G-invariant $\Lambda \subset \mathrm{Maps}(V[1], X)$
  (has a $\Lambda^n V^*$-valued symplectic form)

<u>Theorem</u>: This data produces a dg-Lagrangian $\mathcal{L} \subset \mathrm{Maps}(T[1]\Sigma, X)$ iff

- $\Lambda \subset \mathrm{Maps}(V[1], X)$ is dg-Lagrangian (w.r.t. Q_X)
- $T\Lambda \subset T\,\mathrm{Maps}(V[1], X)$ is a $C^\infty(V[1]) = \Lambda V^*$-submodule

+ we still want ellipticity, i.e. $\alpha \wedge \cdot$ should be acyclic $\forall\, 0 \neq \alpha \in V^*$

Hamiltonian (= BRST/BFV) description



$$W = \mathbb{R} \times W_0$$

$$\Sigma = \mathbb{R} \times \Sigma_0$$

$$V = \mathbb{R} \oplus V_0$$

$X' := \mathrm{Maps}(V_0[1], X)$

$$\Lambda \subset \mathrm{Maps}(\mathbb{R}[1], X') = T^* X' \otimes \Lambda^{n-1} V_0^*$$

ellipticity $\Rightarrow$ Λ given by a dg Lagr. $L \subset X'$
and by a closed function $\mathcal{H} : L \to \Lambda^{n-1} V_0^*$

$\rightsquigarrow$ Hamiltonian picture

phase space $\mathcal{M} = \mathrm{Maps}(T[1]W_0, X)_{\text{b.c. given by } L}$
+ differential from $T[1]W_0$ and X (AKSZ)

hamiltonian $H = \int_{\Sigma_0} \mathcal{H}$ seen as a $n-1$-form on Σ_0

$n=1$ case : $L \subset X = T^*[1]M$, $L = N^*[1]C$, $\mathcal{H}$ of degree 0

$\Rightarrow$ $\mathcal{H}$ a function on C = Hamiltonian mechanics on reduced C

Examples for $n=2$ & conformal structure on Σ

$X = \mathfrak{g}[1]$ (Chern-Simons in the bulk)

generalized metric

Λ given by an ortogonal vector-space decomposition $\mathfrak{g} = C_+ \oplus C_-$

$$\Lambda = \left(V^{*0,1} \otimes C_+[1] \oplus V^{*1,0} \otimes C_-[1]\right) \oplus \Lambda^2 V^* \otimes \mathfrak{g}[1] \subset \Lambda V^* \otimes \mathfrak{g}[1]$$

$$\mathcal{M} = \Omega_{b.c.}(D) \otimes \mathfrak{g}[1] \qquad \Omega^k_{b.c.}(D) = \begin{cases} \Omega^k(D) & \text{if } k \geq 1 \\ \text{vanishing on } S^1 = \partial D & \text{if } k=0 \end{cases}$$

= BRST description of flat $\mathfrak{g}$-connections on $D/$gauge trivial on S^1

$$H(A) = \frac{1}{2}\int_{S^1} \langle a(\sigma), R\,a(\sigma)\rangle\, d\sigma \qquad A|_{S^1} = a(\sigma)\,d\sigma, \quad R: \mathfrak{g} \to \mathfrak{g}$$

the reflection w.r.t. C_+

X an exact Courant algebroid

$\mathcal{M}$ = BRST model of $T^*(LM)$ twisted by closed 3-form

H = a 2d σ-model hamiltonian

Acyclic X's and standalone models

X of a special form : $X = T^*[n]\, T[1]\, Z$

(possibly twisted by a closed n-form)

$$\mathrm{Maps}(V[1], X) = T^* \mathrm{Maps}(V[1], T[1]Z)$$

Λ given by $R \subset \mathrm{Maps}(V[1], T[1]Z)$ and a function $R \to \Lambda^n V^*$

= lagr. density

example : $Z = \mathfrak{g}[1]$ and Yang-Mills

$$R \subset \mathrm{Maps}(V[1], T[1]\mathfrak{g}[1]) = \Lambda V^* \otimes (\mathfrak{g}[1] \oplus \mathfrak{g}[2])$$

$$R = \Lambda V^* \otimes \mathfrak{g}[1] \oplus \Lambda^{\geq 2} V^* \otimes \mathfrak{g}[2]$$

($\Lambda^0 V^* \otimes \mathfrak{g}[1]$ is for gauge symmetry)

+ closed deg. 0 function

$R \to \Lambda^n V^*$

= YM Lagrangian density

$\Lambda^2 V^* \otimes \mathfrak{g}[2] \longrightarrow \Lambda^n V^*$

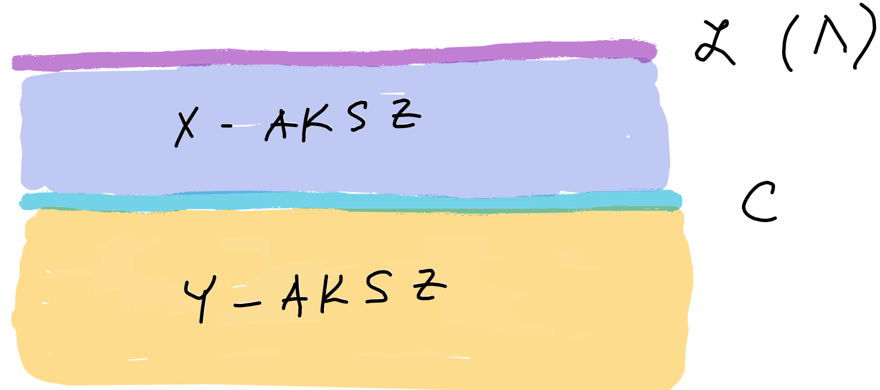
Boundary models to standalone: "dualities"

For a non-acyclic X (and $\Lambda \subset \mathrm{Maps}(V[1], X)$) we get a nontopological model living on the boundary.

Killing (mostly) the bulk: cf SymTFT & quiche [Freed – Moore – Teleman]

$Y = T^*[n]T[1]Z$, $X \simeq$ the reduction of a dg coiso $C \subset Y$

splicing together creates a standalone model



examples: $X = \mathbb{R}^2[2]$ $(n=4)$, $Y \cong T^*[4]T[1]\mathbb{R}[1] = \mathbb{R}[1] \oplus \mathbb{R}^2[2] \oplus \mathbb{R}[3]$

$\rightarrow$ electric/magnetic duality

$X = \mathfrak{g}[1]$, $Y =$ ECA over G/H

$\rightarrow$ Poisson-Lie T-duality (for different H's)

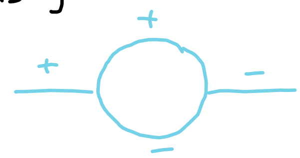
RG flow of $\Lambda \subset \mathrm{Maps}(V[1], X)$

as the first perturbative calculation

Done for $n = 2$ [Pulmann - Š - Youmans]

Generalized Ricci tensor

(moving C_+ within $\mathfrak{g}$)



: $C_+ \to C_-$

In general it is, formally, given by a function on each Λ

$$= -\lim_{\varepsilon \to 0_+} \varepsilon \frac{d}{d\varepsilon} L_{\mathrm{eff}}$$

the divergent (& local) part of the effective action

and... it would be so great to understand it

THANK YOU