Graphs, entanglement and the PPT^2 conjecture

Satvik Singh and Ion Nechita – [arXiv:2010.11891, 2011.03809, 2010.07898]
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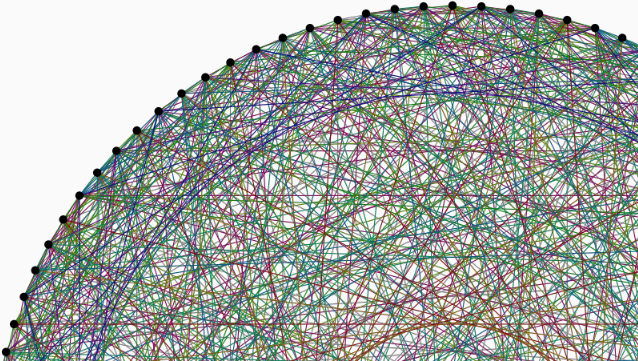

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- In the first half of the talk, we will witness an intriguing connection between two seemingly unrelated mathematical objects: graphs and entangled quantum states, which will allow us to formulate a simple method of detecting entanglement in quantum states.
- In the second half, we will see how the techniques from the first part can be used to tackle the PPT^2 conjecture for a large class of states.

Local diagonal orthogonal invariant (LDOI) states

Definitions

- We consider quantum states $\rho \in \mathcal{M}_d \otimes \mathcal{M}_d$ which enjoy the following local diagonal orthogonal invariance (LDOI) property:

$$\forall O \in \mathcal{DO}_d, \quad \rho = (O \otimes O)\rho(O \otimes O).$$

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- These states admit the following parameterization:

$$\rho_{A,B,C} = \sum_{i,j=1}^{d} A_{ij} |ij\rangle\langle ij| + \sum_{1 \le i \ne j \le d} B_{ij} |ii\rangle\langle jj| + \sum_{1 \le i \ne j \le d} C_{ij} |ij\rangle\langle ji|$$

where $A, B, C \in \mathcal{M}_d$ have equal diagonals, $A \succcurlyeq 0$, $B \geq 0$, $C = C^\dagger$ and $A_{ij}A_{ji} \geq |C_{ij}|^2 \ \forall i,j$. (PPT $\implies$ $C \geq 0$ and $A_{ij}A_{ji} \geq |B_{ij}|^2 \ \forall i,j$)

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- The class of LDOI states is quite rich and contains many prominent names like the Diagonal symmetric states, Werner and Isotropic states, etc.

$3 \otimes 3$ LDOI state

$$\rho_{A,B,C} = \left(\begin{array}{ccc|ccc|ccc}
A_{11} & \cdot & \cdot & \cdot & B_{12} & \cdot & \cdot & \cdot & B_{13} \\
\cdot & A_{12} & \cdot & C_{12} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & A_{13} & \cdot & \cdot & \cdot & C_{13} & \cdot & \cdot \\
\hline
\cdot & C_{21} & \cdot & A_{21} & \cdot & \cdot & \cdot & \cdot & \cdot \\
B_{21} & \cdot & \cdot & \cdot & A_{22} & \cdot & \cdot & \cdot & B_{23} \\
\cdot & \cdot & \cdot & \cdot & \cdot & A_{23} & \cdot & C_{23} & \cdot \\
\hline
\cdot & \cdot & C_{31} & \cdot & \cdot & \cdot & A_{31} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & C_{32} & \cdot & A_{32} & \cdot \\
B_{31} & \cdot & \cdot & \cdot & B_{32} & \cdot & \cdot & \cdot & A_{33}
\end{array}\right)$$

Separability of LDOI states

Recall that $\rho \in \mathcal{M}_d \otimes \mathcal{M}_d$ is called separable if there exists a finite set of vectors $\{|v_k\rangle, |w_k\rangle\}_{k\in I} \subseteq \mathbb{C}^d$ such that $\rho = \sum_{k\in I} |v_k w_k\rangle\langle v_k w_k|$.

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Theorem 1 [SN20a]

For $A, B, C \in \mathcal{M}_d$ with equal diagonals, the state $\rho_{A,B,C}$ is separable if and only if (A, B, C) is *triplewise completely positive* (TCP), i.e. there exists a finite set of vectors $\{|v_k\rangle, |w_k\rangle\}_{k\in I} \subseteq \mathbb{C}^d$ such that

$$A = \sum_{k\in I} |v_k \odot \overline{v_k}\rangle\langle w_k \odot \overline{w_k}| \quad B = \sum_{k\in I} |v_k \odot w_k\rangle\langle v_k \odot w_k|$$

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Proposition 1 [SN20a]

(A, A, A) is TCP $\iff$ A is CP (i.e $A = ZZ^\top$ for some $d \times r$ matrix $Z \succcurlyeq 0$). In particular, deciding membership in the TCP cone inherits the NP-hardness of deciding membership in the CP cone.

Triangle-free states

Definitions

- For $\rho \in \mathcal{M}_d \otimes \mathcal{M}_d$, define $A \in \mathcal{M}_d$ entrywise as $A_{ij} = \langle ij|\rho|ij\rangle$. We now associate a d-vertex graph to ρ as follows. Two distinct vertices $i \neq j$ form an edge (i.e. they are connected) if both A_{ij} and A_{ji} are non-zero.

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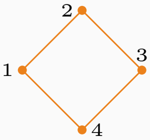
- Example:

$G(A) =$ 

$$A = \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} \\ a_{21} & 2 & a_{23} & 0 \\ 0 & a_{32} & 2 & a_{34} \\ a_{41} & a_{42} & a_{43} & 2 \end{pmatrix}$$

Entanglement detection in Δ-free states

Separability of Δ-free LDOI states

Theorem 2 [Sin20]

If $\rho_{A,B,C}$ is Δ-free and separable, then $M(B) \geq 0$ and $M(C) \geq 0$.

where, for $B \in \mathcal{M}_d$, the comparison matrix is defined entrywise as follows:

$$M(B)_{ij} = \begin{cases} |B_{ij}|, & \text{if } i = j \\ -|B_{ij}|, & \text{otherwise} \end{cases}$$

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- Step 2: Use Δ-freeness of $G(A)$ to show that $\# \operatorname{supp}(v_k \odot w_k) \leq 2 \ \forall k$.
- Step 3: $B = \sum_{k \in I} |v_k \odot w_k\rangle\langle v_k \odot w_k| \implies M(B) \geq 0$.

Primary entanglement detection test

Theorem 3 [Sin20]

Let $\rho \in \mathcal{M}_d \otimes \mathcal{M}_d$ be Δ-free. Let $A, B, C \in \mathcal{M}_d$ be defined entrywise as

$$A_{ij} = \langle ij|\rho|ij\rangle, \quad B_{ij} = \langle ii|\rho|jj\rangle, \quad C_{ij} = \langle ij|\rho|ji\rangle \qquad \text{for } i, j \in [d].$$

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Proof. Project onto the LDOI subspace: $\rho_{A,B,C} = \mathbb{E}_O[(O \otimes O)\rho(O \otimes O)]$. □

Consequences and discussion

Beauty of Δ-free entanglement

- Given any Δ-free (cyclic) graph on d vertices, an associated family of Δ-free PPT-entangled states with d^2 real parameters can be constructed.

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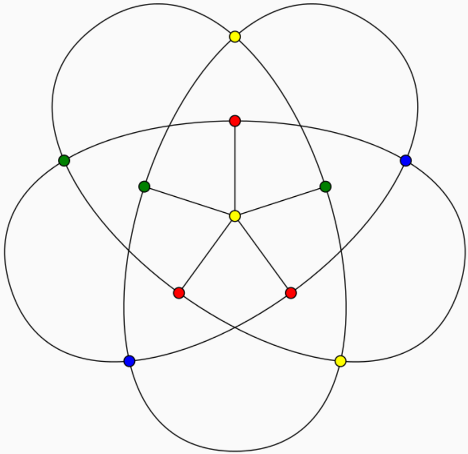

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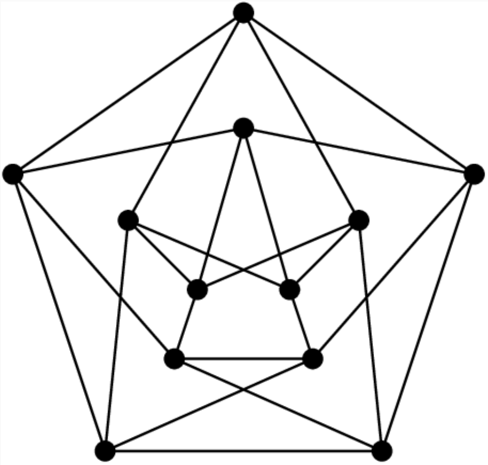

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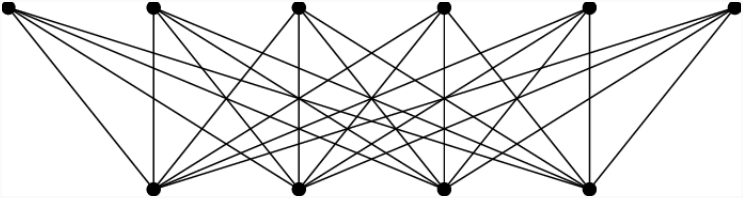

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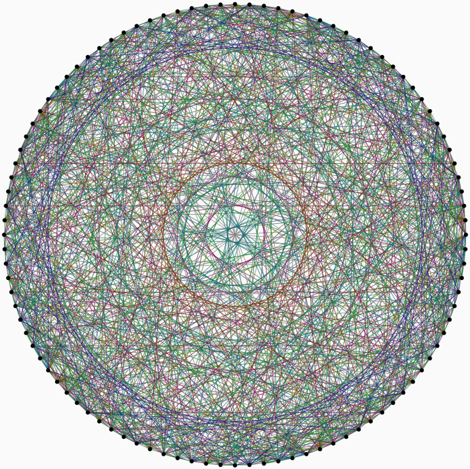

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- For instance, using the above test, it is possible to construct $\sim 10^{10}$ PPT-entangled families of states in a paltry $15 \otimes 15$ system.
- Because of this sheer diversity, the traditional methods for entanglement detection get crippled in the regime of Δ-free states. In contrast, the simplicity of Δ-free entanglement detection cannot be overstated.

Can we say something about the converse of Theorem 2?

What we know

If a given LDOI state $\rho_{A,B,C}$ is separable and $\{|v_k\rangle, |w_k\rangle\}_{k\in I}$ form a TCP decomposition of (A, B, C), then

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What we want to know

Given an LDOI state $\rho_{A,B,C}$, does

$$M(B), M(C) \geq 0 \implies \rho_{A,B,C} \text{ is separable?}$$

- Converse only holds for a special subclass of LDOI states which are of the form given below, where $A, B \in \mathcal{M}_d$ have equal diagonals, $A \succcurlyeq 0$ and $B \geq 0$. PPT condition additionally implies that $A_{ij}A_{ji} \geq |B_{ij}|^2 \; \forall i,j$.

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- These states enjoy the following *conjugate local diagonal unitary invariance* (CLDUI) property

$$\forall U \in \mathcal{DU}_d, \quad \rho_{A,B} = (U \otimes U^\dagger)\rho_{A,B}(U^\dagger \otimes U)$$

Factor widths and the converse of Theorem 2

Theorem 4 [JM19, SN20b]

The state $\rho_{A,B}$ is separable if and only if (A, B) is *Pairwise Completely Positive* (PCP), i.e. there exist vectors $\{|v_n\rangle, |w_n\rangle\}_{n\in I}$ such that

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Theorem 5 [SN20b]

Let $\rho_{A,B}$ be a PPT CLDUI state. Then,

$$M(B) \geq 0 \iff (A, B) \text{ is PCP with factor width 2} \implies \rho_{A,B} \text{ is separable.}$$

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Conjecture (Form 1) [Chr12]

Let $\rho, \sigma \in \mathcal{M}_d \otimes \mathcal{M}_d$ be arbitrary bipartite PPT states. Then, $\mathrm{Tr}_C[(\rho \otimes \sigma)(\mathbb{I}_A \otimes |\psi_C^+\rangle\langle\psi_C^+| \otimes \mathbb{I}_B)]$ is separable.

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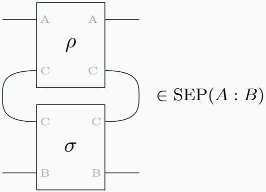

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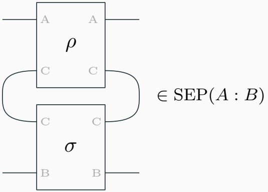

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Conjecture (Form 2) [Chr12]

Let $\Phi_1, \Phi_2 : \mathcal{M}_d \to \mathcal{M}_d$ be arbitrary PPT (CP + *co*CP) linear maps. Then, $\Phi_1 \circ \Phi_2$ is entanglement breaking.

Progress on the proof

- The conjecture holds for $d = 2$ (trivially) and $d = 3$ [CMHW18, CYT19].

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- The conjecture holds for Gaussian maps [CMHW18].

CLDUI states $\leftrightarrow$ CDUC maps

- Define $\mathcal{T}_d(\mathbb{C}) := \{\Phi : \mathcal{M}_d \to \mathcal{M}_d \text{ linear}\}$ and let $J : \mathcal{M}_d \otimes \mathcal{M}_d \to \mathcal{T}_d(\mathbb{C})$ denote the Choi-Jamiołkowski isomorphism. Let $J(\rho_{A,B}) = \Phi_{A,B}$. Then,

$$\begin{aligned} \Phi_{A,B} : \mathcal{M}_d &\to \mathcal{M}_d \\ X &\mapsto \operatorname{diag}(A\,|\operatorname{diag} X\rangle) + (B - \operatorname{diag} B) \odot X \end{aligned}$$

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- These maps possess a special diagonal unitary covariance property:

$$\forall U \in \mathcal{DU}_d, \forall X \in \mathcal{M}_d : \;\; \Phi_{A,B}(UXU^*) = U\Phi_{A,B}(X)U^*$$

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- Define $\mathcal{T}_d(\mathbb{C}) \coloneqq \{\Phi : \mathcal{M}_d \to \mathcal{M}_d \text{ linear}\}$ and let $J : \mathcal{M}_d \otimes \mathcal{M}_d \to \mathcal{T}_d(\mathbb{C})$ denote the Choi-Jamiołkowski isomorphism. Let $J(\rho_{A,B}) = \Phi_{A,B}$. Then,

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- These maps possess a special diagonal unitary covariance property:

$$\forall U \in \mathcal{DU}_d, \forall X \in \mathcal{M}_d : \quad \Phi_{A,B}(UXU^*) = U\Phi_{A,B}(X)U^*$$

- One can immediately observe that the Choi map is of the above form:

$$\begin{aligned} \Phi_{\text{choi}} : \mathcal{M}_3 &\to \mathcal{M}_3 \\ X &\mapsto \begin{pmatrix} X_{11} + X_{33} & -X_{12} & -X_{13} \\ -X_{21} & X_{11} + X_{22} & -X_{23} \\ -X_{31} & -X_{32} & X_{22} + X_{33} \end{pmatrix} \end{aligned}$$

The PPT^2 conjecture holds for CLDUI states

Theorem 5 [SN20b]

Let $\rho_{A,B}$ be a PPT CLDUI state. Then,

$$M(B) \geq 0 \iff (A, B) \text{ is PCP with factor width 2} \implies \rho_{A,B} \text{ is separable.}$$

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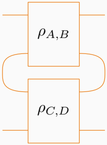
- Step 1: 

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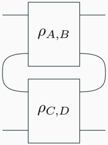
- Step 1: 

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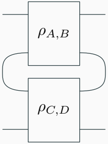
- Step 1: 

$= \rho_{X,Y}$; where $(X, Y) = (AC, B \odot D + \mathrm{diag}(AC - B \odot D))$.
- Step 2: Show that the CLDUI state $\rho_{X,Y}$ is again PPT.
- Step 3: Show that $M(Y) \geq 0$ and apply Theorem 5 to conclude the proof.

To gain some insight on the structure of Y, let us look at the 3×3 case:

$$Y = \begin{pmatrix} A_{11}C_{11} + A_{12}C_{21} + A_{13}C_{31} & B_{12}D_{12} & B_{13}D_{13} \\ B_{21}D_{21} & A_{21}C_{12} + A_{22}C_{22} + A_{23}C_{32} & B_{23}D_{23} \\ B_{31}D_{31} & B_{32}D_{32} & A_{31}C_{13} + A_{32}C_{23} + A_{33}C_{33} \end{pmatrix}$$

$$= \text{diag}(A \odot C) + \begin{pmatrix} A_{12}C_{21} & B_{12}D_{12} & 0 \\ B_{21}D_{21} & A_{21}C_{12} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} A_{13}C_{31} & 0 & B_{13}D_{13} \\ 0 & 0 & 0 \\ B_{31}D_{31} & 0 & A_{31}C_{13} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & A_{23}C_{32} & B_{23}D_{23} \\ 0 & B_{32}D_{32} & A_{32}C_{23} \end{pmatrix}$$

Infact, a similar decomposition holds for the general $d \times d$ case as well:

$$\boxed{Y = \text{diag}(A \odot C) + \sum_{1 \le i < j \le d} \begin{pmatrix} A_{ij}C_{ji} & B_{ij}D_{ij} \\ B_{ji}D_{ji} & A_{ji}C_{ij} \end{pmatrix}_{i,j \in [d]}}$$

Since $A_{ij}A_{ji} \ge |B_{ij}|^2, C_{ij}C_{ji} \ge |D_{ij}|^2 \; \forall i, j$ and $A, C \succcurlyeq 0$, the above decomposition shows that $M(Y) \ge 0$, which completes the proof. $\square$

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- In particular, we don't know if the conjecture holds for LDOI states $\rho_{A,B,C}$.

Thank you!

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