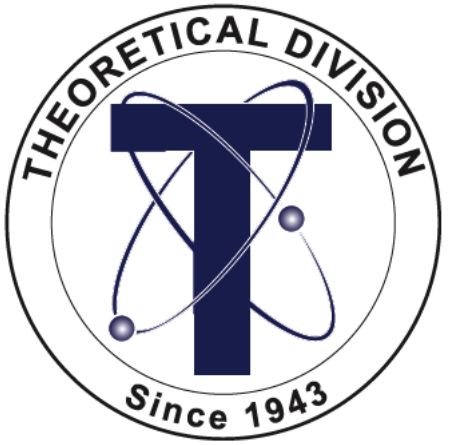
Hamiltonian simulation in the low energy subspace

Burak Sahinoglu

with Rolando D. Somma



Check it @ arXiv:2006.02660 **[quant-ph]**

Watch the tutorial by Andrew Childs:
https://www.youtube.com/watch?v=M0e5gkf7QSQ

LA-UR-20-27396

Hamiltonian Simulation

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  Given a Hamiltonian H, "simulate" the time evolution:

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- Main approaches for general time evolution:

  1- Product formulas

  (e.g., Trotter-Suzuki)

  2- Taylor series method

  3- Quantum signal processing

It's a well-developed field with many important works! Hard to capture even the milestones here. ---> Check the tutorial by Childs!

Hamiltonian Simulation: Important physical scenarios

- Simulating quantum field theories:

  The interesting and reliable physics takes place in a low-energy subspace, which is way below the energy cutoff of the regularized theory.

  QFTs with bosonic dofs suffer from infinite-norm problem.

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- Simulating many-body quantum dynamics:

  Adiabatic evolution takes place in the low-energy subspace.

  Many interesting Hamiltonians: Spin models such as the Heisenberg models, etc.

- Believed to be one of the main meaningful applications of quantum computers.

Hamiltonian Simulation in the low energy subspace

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  Given a Hamiltonian H, simulate the time evolution of a state $|\Psi\rangle$ which lives in a low energy subspace:

$$e^{-iHt}|\Psi\rangle$$

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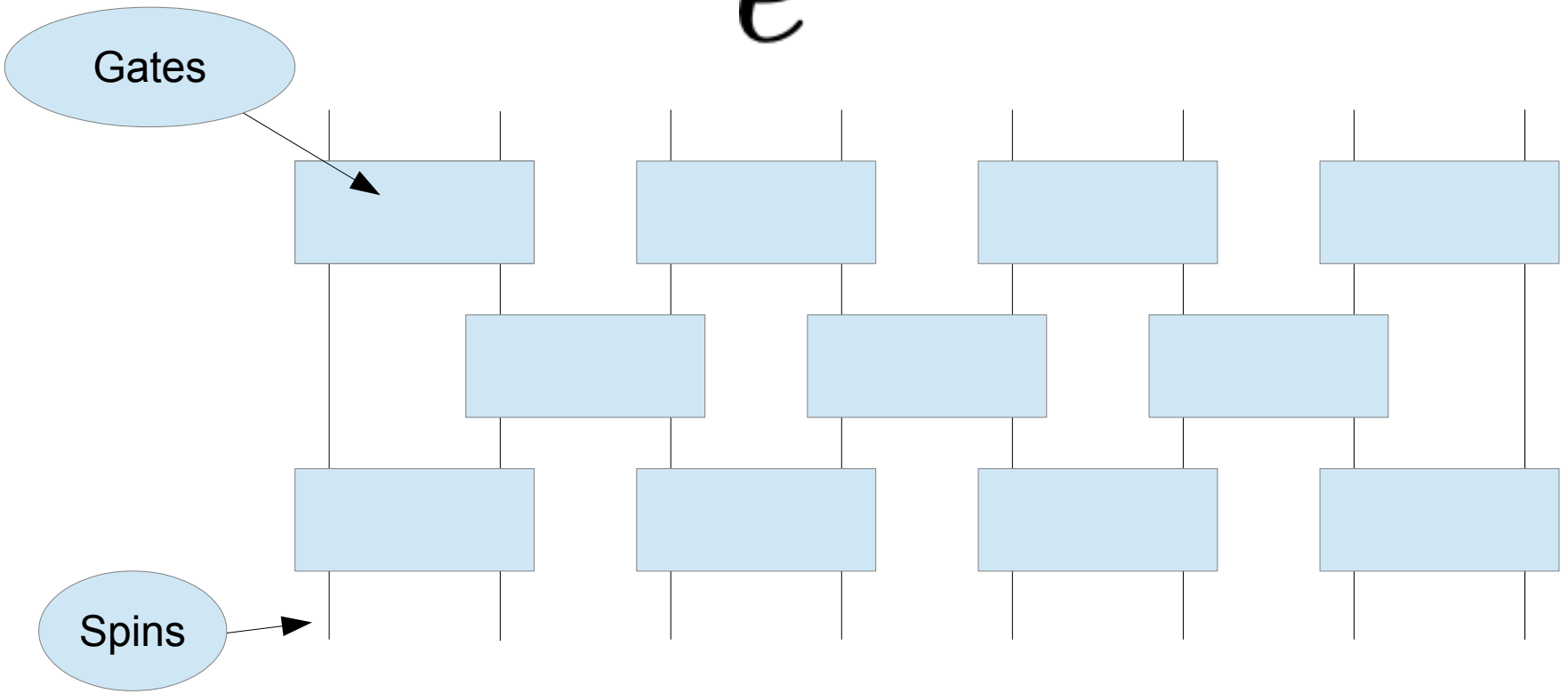
We expect an improvement in the performance if/when during the evolution under the product formula, the state is not supported in the high-energy subspace!

What do we mean by "simulation"?

Simulation means that we want to find a quantum circuit W that approximates the time evolution operator

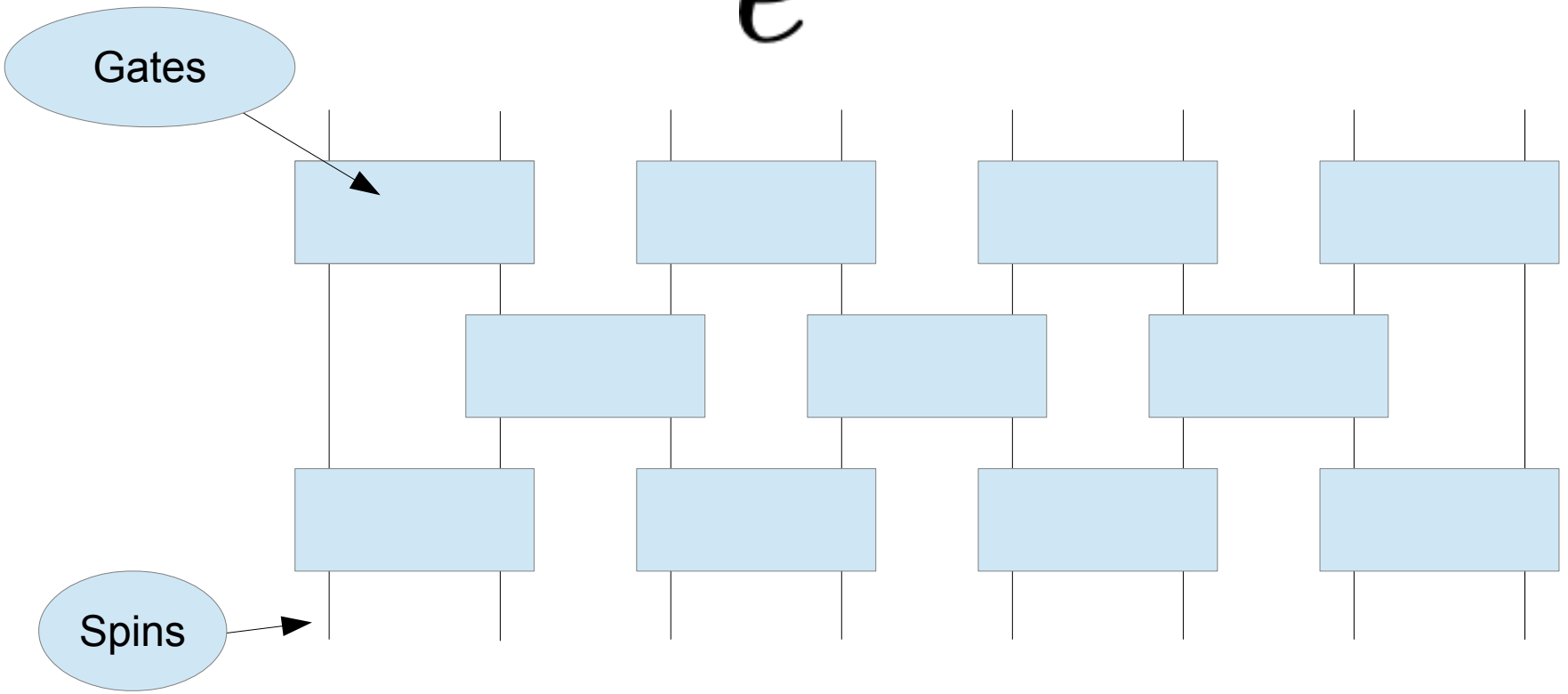
$$e^{-iHt}$$



What is the goal?

Use/find the least possible number of gates in W, such that a desired accuracy is met for approximating the operator

$$e^{-iHt}$$



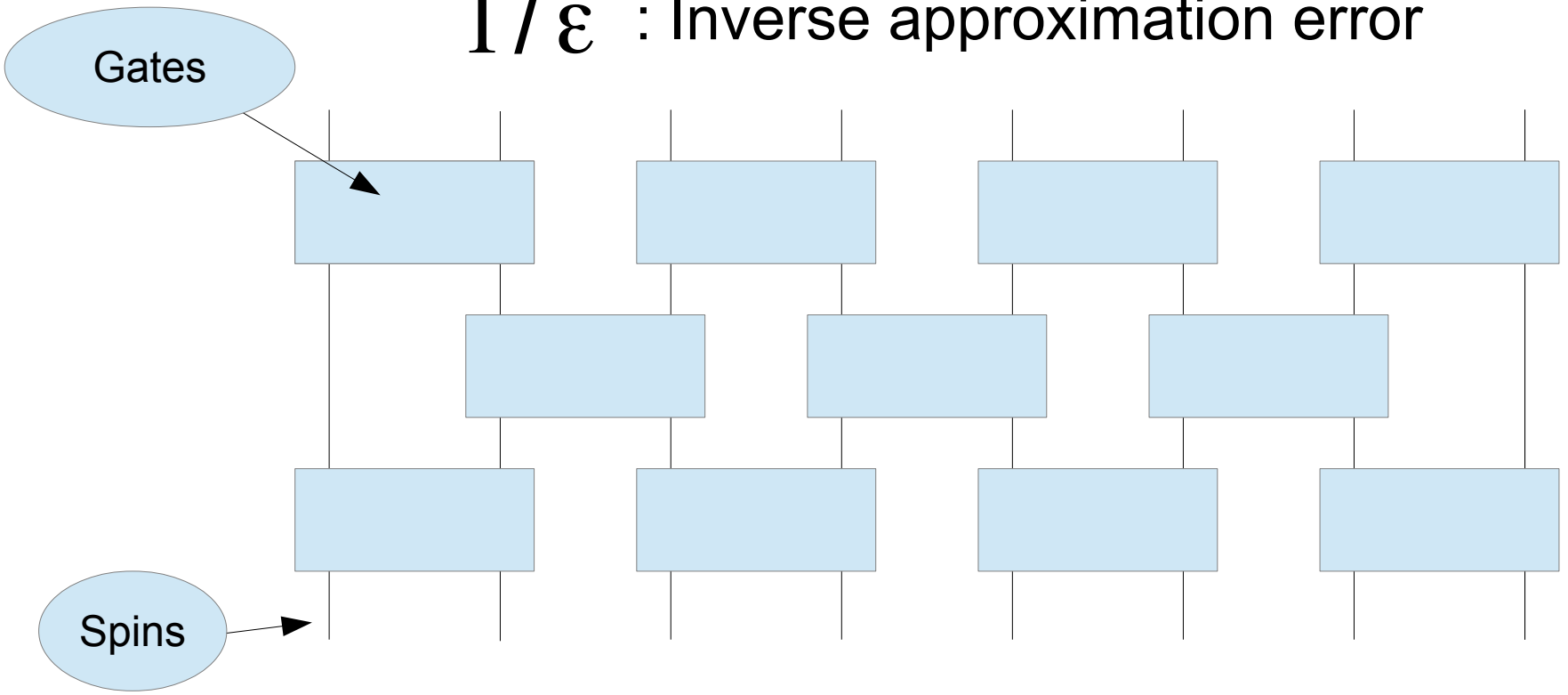
More precisely

Find a quantum circuit W with the smallest number of gates (e.g., linear or polylog) in

t : total evolution time

N : total number of spins

$1/\varepsilon$: Inverse approximation error



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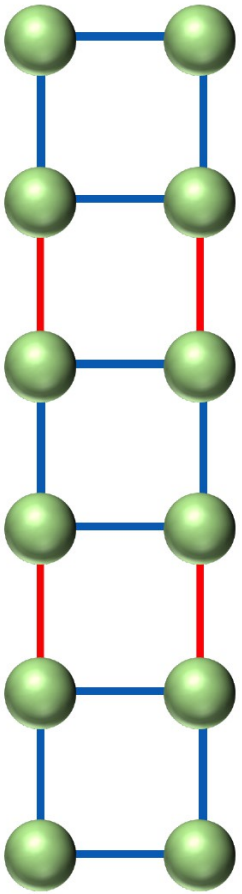
A small scale simulation – Heisenberg model

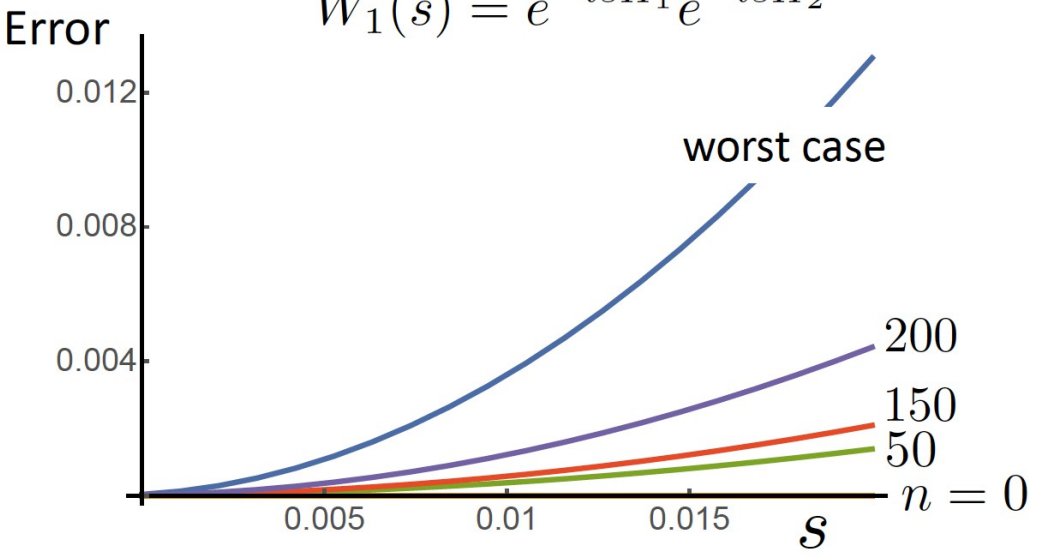
$$H = H_1 + H_2$$



$$H = -\sum_{\langle i,j \rangle} X_i X_j + Y_i Y_j + Z_i Z_j$$

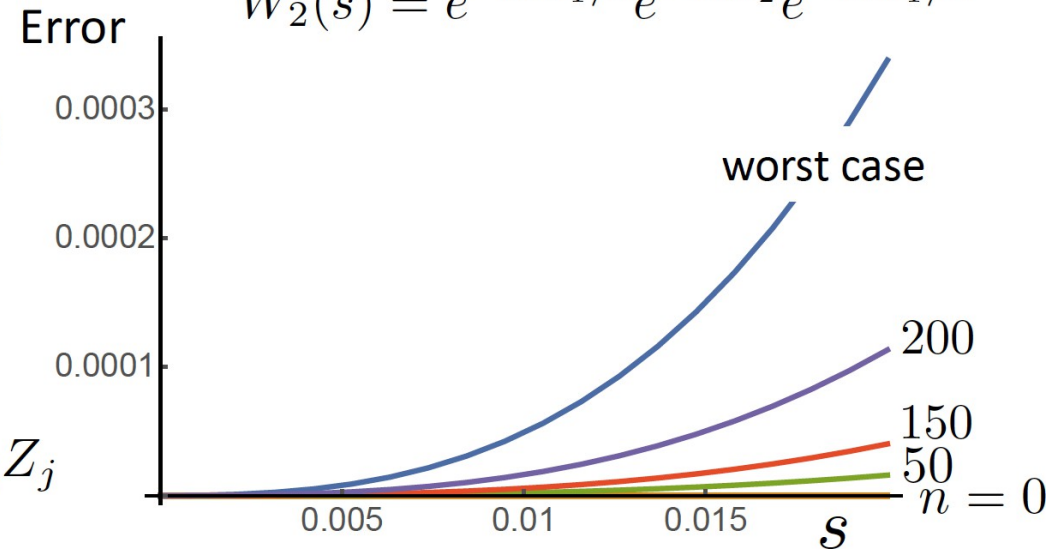
(a)

$$W_1(s) = e^{-isH_1} e^{-isH_2}$$



(b)

$$W_2(s) = e^{-isH_1/2} e^{-isH_2} e^{-isH_1/2}$$



Why do we bother going beyond small scale simulations?

- An advantage in a small scale simulation does not guarantee anything about any larger scale simulation.

- A large scale simulation is (believed to be) not possible without a quantum computer. We do not have any, yet! So, analytical bounds are needed.

- Simulations are done for a specific Hamiltonian, each time. However, we believe that the low-energy simulation is less costly for a broad class of models.

Hamiltonian Simulation in the low energy subspace

- The problem:

  Given a Hamiltonian H , simulate the time evolution of a state $|\Psi\rangle$ which lives in a low energy subspace:

$$e^{-iHt}|\Psi\rangle$$

- Main approaches for general time evolution:

  1- Product formulas

  (e.g., Trotter-Suzuki)

  2- Taylor series method

  3- Quantum signal processing

We expect an improvement in the performance if/when during the evolution under the product formula, the state is not supported in the high-energy subspace!

A possible way of treating the low-energy simulation problem

- Find a low-energy effective Hamiltonian, such that the dynamics in the low-energy subspace is faithful to the original one.

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- Find a low-energy effective Hamiltonian, such that the dynamics in the low-energy subspace is faithful to the original one.

Problems:

1- The Hamiltonian gets non-local.

2- There is no general and efficient procedure for finding these effective Hamiltonians.

Our way of treating the low-energy simulation problem

- Apply the product formula with the original Hamiltonian, and study how faithful it is to an exact low-energy evolution.

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  Problems:

  1- The product formula makes the state leak towards higher energies.

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- Apply the product formula with the original Hamiltonian, and study how faithful it is to an exact low-energy evolution.

  Problems:

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  2- Locality is harder to exploit.

- At the end, things work out just fine so that we can prove a polynomial improvement.

General Product Formulas

- Hamiltonian is decomposed:

$$H = \sum_l H_l$$

- A simple case: $H = H_{\text{even}} + H_{\text{odd}}$

- Ideally, we choose this decomposition such that the Hamiltonian terms in each of the terms H_l are commuting.

General Product Formulas

- A p-th order product formula implements a product of

  $e^{-iH_l s}$ with a particular sequence of H_l's, i.e.,

  $$W_p(s) = e^{-is_q H_{l_q}} \cdots e^{-is_2 H_{l_2}} e^{-is_1 H_{l_1}}$$

  such that

  $$\|U(s) - W_p(s)\| = \mathcal{O}((L\|H_l\||s|)^{p+1})$$

- This is repeated for $r = t/s$ -many times for the complete simulation, hence the total error is

  $$\|U(t) - (W_p)^r\| = r \cdot \mathcal{O}(L\|H_l\|t/r)^{p+1}$$

- r is known as the Trotter number – it is our complexity measure.

Our result: Product formulas at low energies

- A p-th order product formula implements a product of $e^{-iH_l s}$ with a particular sequence of H_l 's. And this is repeated for $r = t/s$ -many times for the complete simulation:

$$W_p(s) = e^{-is_q H_{l_q}} \cdots e^{-is_2 H_{l_2}} e^{-is_1 H_{l_1}}$$

such that

$$\|(U(s) - W_p(s))\Pi_{\leq \Delta}\| = \mathcal{O}((L\|H_l\||s|)^{p+1})$$

Effective norm

Δ'

- The total error is

$$\|(U(t) - (W_p(s))^r)\Pi_{\leq \Delta}\| = r \cdot \mathcal{O}((L\|H_l\||t/r|)^{p+1})$$

Δ'

Our (simplified) results: The complexity ($= r$)

$$p = 1, 2, 3; \quad \tau = Jt \quad \text{and} \quad k, d, \Delta = \mathcal{O}(1)$$

Order	Previous result	Low-energy simulation
$p = 1$	$\mathcal{O}(\frac{\tau^2 N}{\varepsilon})$	$\tilde{\mathcal{O}}(\frac{\tau^2}{\varepsilon}) + \mathcal{O}(\frac{\tau^{4/3} N^{2/3}}{\varepsilon^{1/3}})$
$p = 2$	$\mathcal{O}(\frac{\tau^{3/2} N^{1/2}}{\varepsilon^{1/2}})$	$\tilde{\mathcal{O}}(\frac{\tau^{3/2}}{\varepsilon^{1/2}}) + \mathcal{O}(\frac{\tau^{6/5} N^{3/5}}{\varepsilon^{1/5}})$
$p = 3$	$\mathcal{O}(\frac{\tau^{4/3} N^{1/3}}{\varepsilon^{1/3}})$	$\tilde{\mathcal{O}}(\frac{\tau^{4/3}}{\varepsilon^{1/3}}) + \mathcal{O}(\frac{\tau^{8/7} N^{4/7}}{\varepsilon^{1/7}})$

Use locality

Use low energy condition

A theory of Trotter error by Childs,Su, Tran, Wiebe, Zhu
@arXiv:1912.08854 [quant-ph]

The structure of the Hamiltonian

- The Hamiltonian is a sum of *k*-local terms acting on *N* spins:

$$H = \sum_{i=1}^{N'} h_i, \quad h_i \geq 0, \quad \|h_i\| \leq J$$

- Product formulas group the Hamiltonian terms:

$$H = \sum_{l=1}^{L} H_l$$

- H is degree-*d* if every spin is included in at most d-many Hamiltonian terms.
- Each H_l is a sum of *M* hamiltonian terms:

$$N' = ML \leq dN$$

For simplicity: nearest neighbor

- The Hamiltonian is a sum of 2-local terms acting on *N* spins:

$$H = \sum_{i=1}^{N'} h_i, \quad h_i \geq 0, \quad \|h_i\| \leq J$$

- Product formulas group the Hamiltonian terms: $L = 2$

$$H = H_{\text{even}} + H_{\text{odd}}$$

- H is degree-2 if every spin is included in at most 2-many local Hamiltonian terms.
- $H_{\text{even}}, H_{\text{odd}}$ both are a sum of *N/2* many hamiltonian terms.

  The terms in $H_{\text{even}}, H_{\text{odd}}$ are chosen to be commuting.

Our result: Product formulas at low energies

- A *p*-th order product formula on a state in the low energy subspace, namely $|\Psi\rangle \in \Pi_{\leq \Delta}$ comes with an error

$$\|(U(s) - W_p(s))\Pi_{\leq \Delta}\| = \mathcal{O}((L\Delta' s)^{p+1})$$

where

$$\Delta' = \Delta + \beta_1 J \log(\beta_2/(J|s|)) + \beta_3 J^2 N|s|$$

- Note that s is tried to be chosen as large as possible!
- Hence the total error is

$$r \cdot \mathcal{O}((L\Delta' s)^{p+1})$$

Our (simplified) result: The complexity ($= r$)

$$r = \tilde{\mathcal{O}}\left(\frac{(t(\Delta+J))^{1+\frac{1}{p}}}{\varepsilon^{\frac{1}{p}}}\right) + \mathcal{O}\left(\frac{(tJ\sqrt{N})^{1+\frac{1}{2p+1}}}{\varepsilon^{\frac{1}{2p+1}}}\right)$$

Comparison for $p = 1, 2, 3$:

Order	Previous result	Low-energy simulation
$p=1$	$\mathcal{O}(\frac{\tau^2 N}{\varepsilon})$	$\tilde{\mathcal{O}}(\frac{\tau^2}{\varepsilon}) + \mathcal{O}(\frac{\tau^{4/3}N^{2/3}}{\varepsilon^{1/3}})$
$p=2$	$\mathcal{O}(\frac{\tau^{3/2}N^{1/2}}{\varepsilon^{1/2}})$	$\tilde{\mathcal{O}}(\frac{\tau^{3/2}}{\varepsilon^{1/2}}) + \mathcal{O}(\frac{\tau^{6/5}N^{3/5}}{\varepsilon^{1/5}})$
$p=3$	$\mathcal{O}(\frac{\tau^{4/3}N^{1/3}}{\varepsilon^{1/3}})$	$\tilde{\mathcal{O}}(\frac{\tau^{4/3}}{\varepsilon^{1/3}}) + \mathcal{O}(\frac{\tau^{8/7}N^{4/7}}{\varepsilon^{1/7}})$

$p = 1, 2, 3; \quad \tau = Jt \quad \text{and} \quad k, d, \Delta = \mathcal{O}(1)$

An important ingredient

- Leakage to high energies by partial evolution operators is exponentially suppressed in energy difference, i.e., there's a sort of locality principle in the energy spectrum.

$$\left\|\Pi_{>\Lambda'} e^{-isH_{\text{even/odd}}}\Pi_{\leq\Lambda}\right\| \leq e^{-\lambda(\Lambda'-\Lambda)}\left(e^{\alpha|s|N}-1\right)$$

$$\lambda = 1/(8J), \alpha = eJ/2$$

- This result comes from a previous result by Arad, Kuwahara, Landau:

$$\left\|\Pi_{>\Lambda'} A \Pi_{\leq\Lambda}\right\| \leq \|A\| \cdot e^{-\lambda(\Lambda'-\Lambda-2R)}$$

Overview of the derivation-1

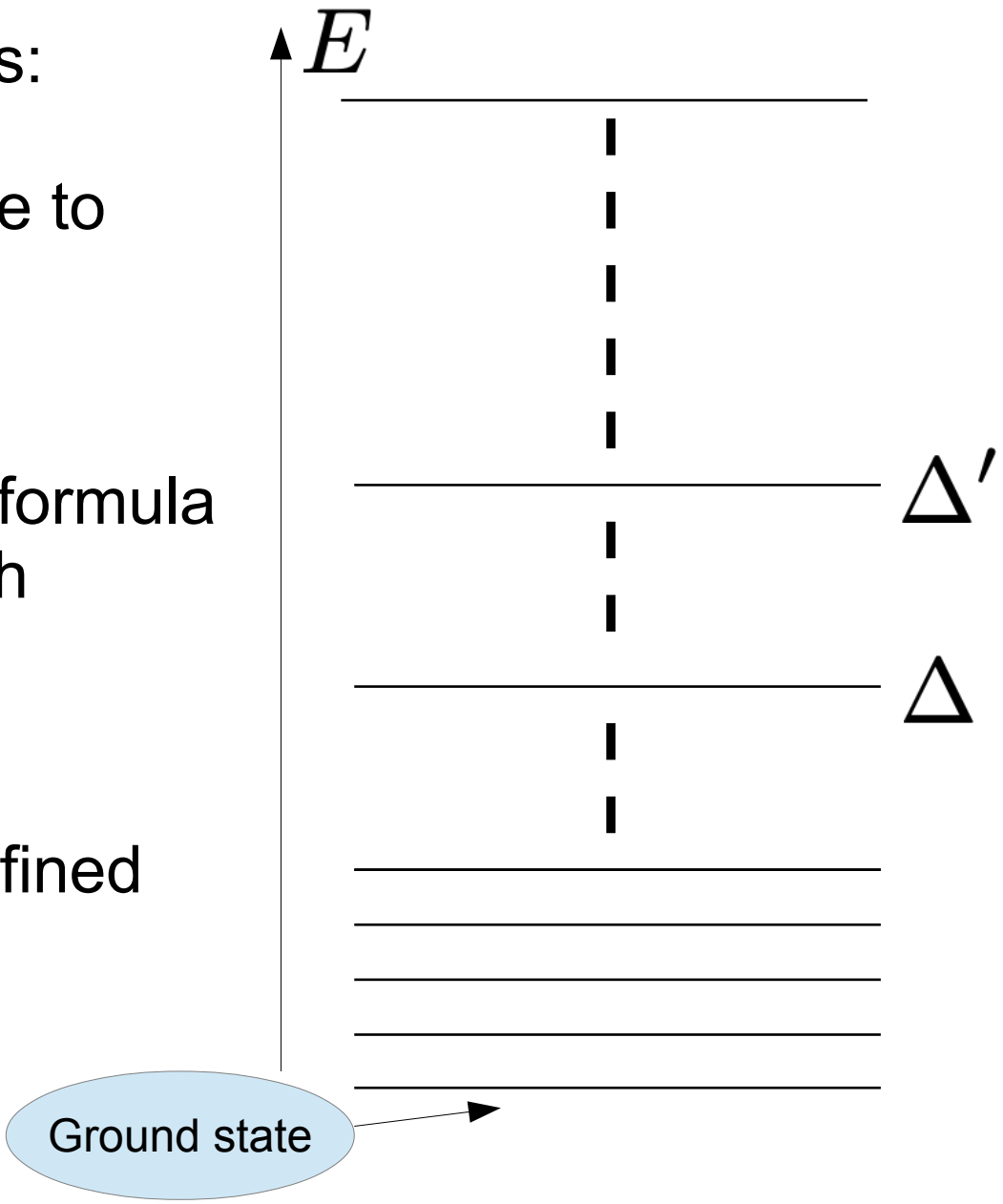
- There are two types of errors:

  1- Error coming from leakage to high energy subspace.

  2- Error coming from the commutators in the product formula written for time evolution with effective operators.

- Let O be an operator. The "effective operator" $\bar{O}$ is defined as

$$\bar{O} := \Pi_{\leq \Delta'} O \Pi_{\leq \Delta'}$$



Overview of the derivation-2

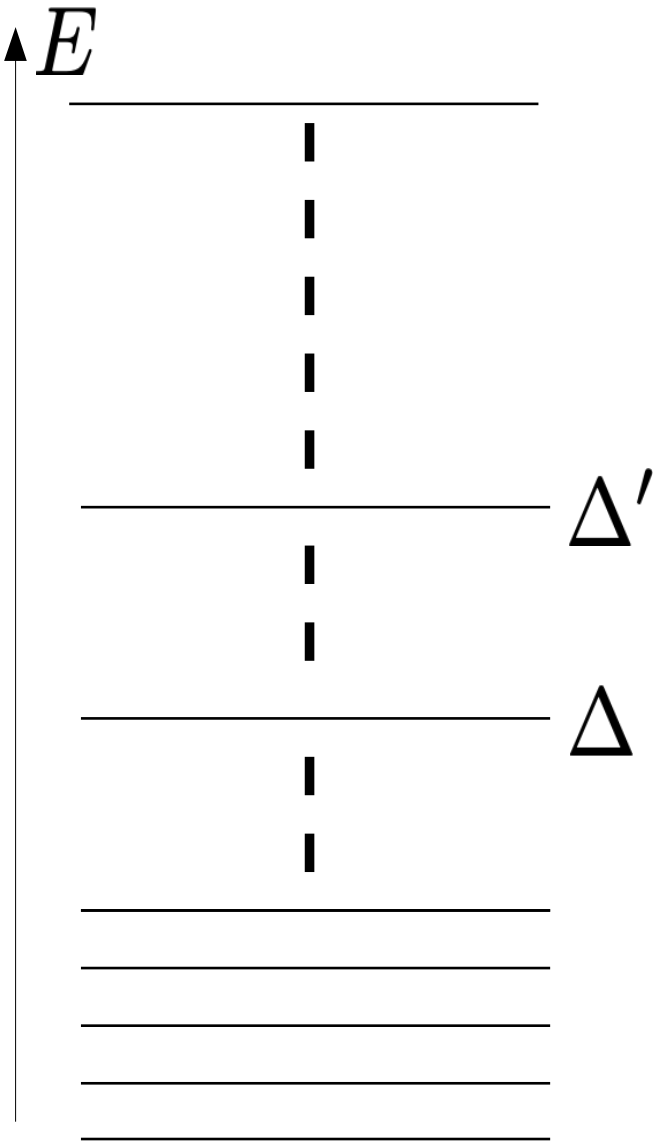
- There are two types of errors:

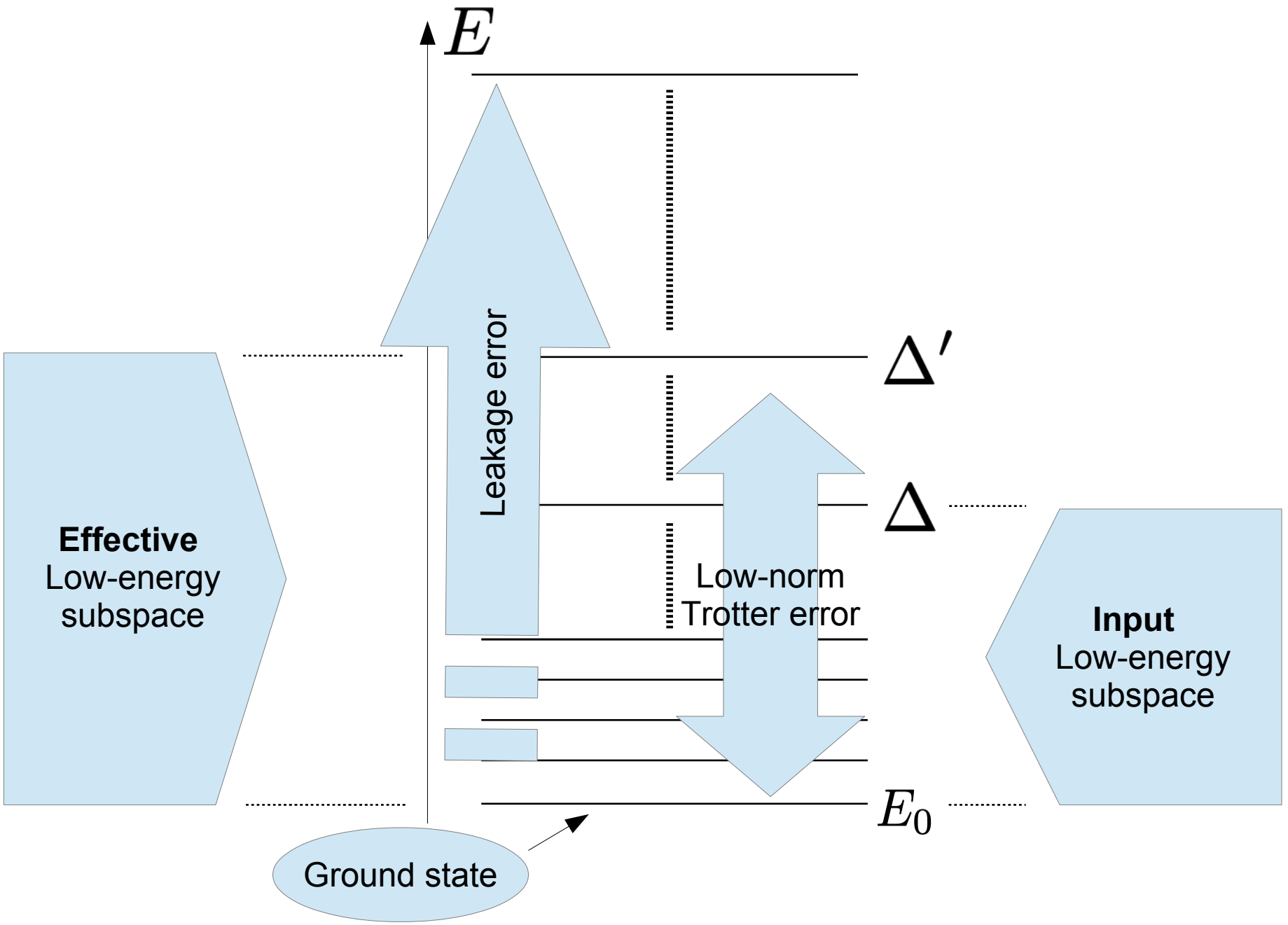
  1- Error coming from leakage to high energy subspace:

$$\|(\bar{W}_p(s) - W_p(s))\Pi_{\leq\Delta}\| \leq \delta(\Delta')$$

  2- Error coming from the commutators in the product formula written for time evolution with effective operators.

$$\|(U(s) - \bar{W}_p(s))\Pi_{\leq\Delta}\| \leq \epsilon(\Delta')$$




E
Leakage error
Δ'
Δ
Low-norm
Trotter error
Effective
Low-energy
subspace
Input
Low-energy
subspace
E_0
Ground state

Low energy simulation errors

$$\|U(s) - W_p(s)\Pi_{\leq\Delta}\| \leq \|U(s) - \bar{W}_p(s)\Pi_{\leq\Delta}\| + \|\bar{W}_p(s) - W_p(s)\Pi_{\leq\Delta}\|$$

Low-norm Trotter error

$$\epsilon(\Delta') \approx \delta(\Delta')$$

Leakage error

$$\|(U(s) - \bar{W}_p(s))\Pi_{\leq\Delta}\| \leq \epsilon(\Delta') \qquad \|(\bar{W}_p(s) - W_p(s))\Pi_{\leq\Delta}\| \leq \delta(\Delta')$$

Standard Trotter error with low norm

$$\epsilon(\Delta') = \gamma(\Delta'|s|)^{p+1}$$

Exponential supression of leakage error

$$\delta(\Delta') = e^{-\alpha(\Delta'-\Delta)/J-\beta|s|JN-\eta}$$

Find the optimal Δ' such that

$$\epsilon(\Delta') \approx \delta(\Delta')$$

Standard Trotter error with low norm

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$p = 1, 2, 3; \quad \tau = Jt \quad \text{and} \quad k, d, \Delta = \mathcal{O}(1)$

Remarks

- Product formulas' complexity are improved in some cases (small p, and/or large t and/or small error) when the low-energy dynamics is considered.
- Childs et.al. make use of locality in the general case. This gives them a lot of leverage. We can not do it because a priori the effective Hamiltonians are completely nonlocal.
- The assumption of $h_i \geq 0$ seems crucial for the results. There are vast amount of models for which this can be satisfied in such a way that $\Delta' \ll \|H\|$.
- Low energy simulation with Taylor series method improves by a log factor (mild improvement).
- Our results also improves upon Taylor series method and quantum signal processing when the error is constant.
- There are more questions and details in the paper.

Remarks

- Our results show the most improvement when

$$\Delta, k, d = O(1)$$

  More improvements for cases beyond these?

  How to make use of locality in our case?

- Important applications:

  Quantum field theories (bosonic systems) - $\|H\| = \infty$

  Frustration free systems - Fully applies

  Adiabatic evolution

- Where can we apply similar ideas?

  Check recent works

  by Tran, Su, Carney, Taylor @ arXiv:2006.16248 [quant-ph] and

  by Su, Huang, Campbell @arXiv:2012.09194 [quant-ph] – **Friday 7 pm @QIP**