AN OPTIMAL SEPARATION OF RANDOMIZED AND QUANTUM QUERY COMPLEXITY

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Central open problem

How much faster can quantum computers be than classical?

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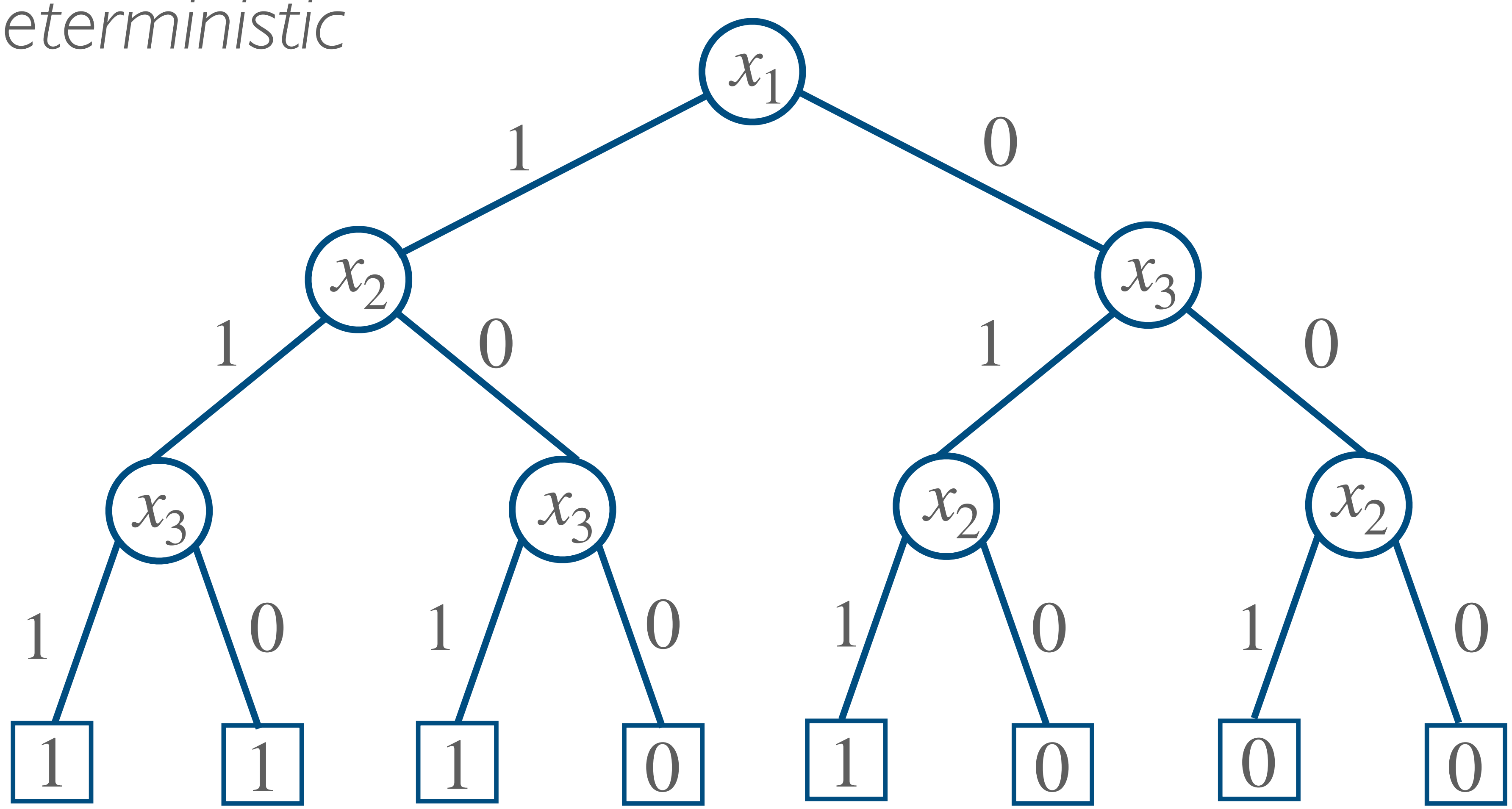
Most research focuses on the query model. e.g.

Deutsch-Jozsa's algorithm	Bernstein-Vazirani's algorithm
Simon's algorithm	Shor's factoring algorithm
Grover's search	……

Query complexity

Deterministic



$$T : \{0,1\}^n \rightarrow \{0,1\}$$

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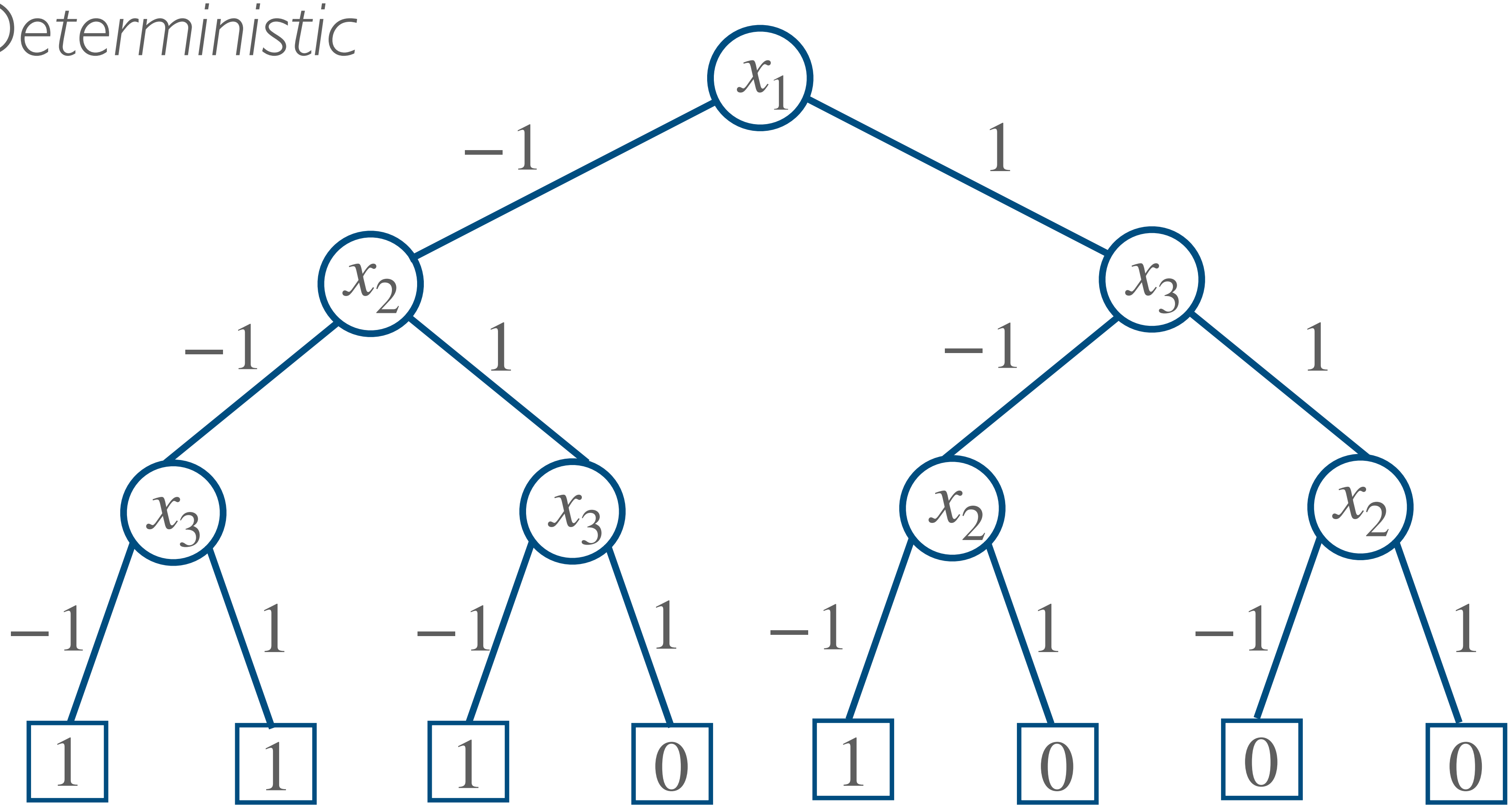

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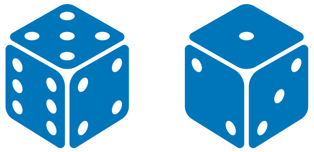
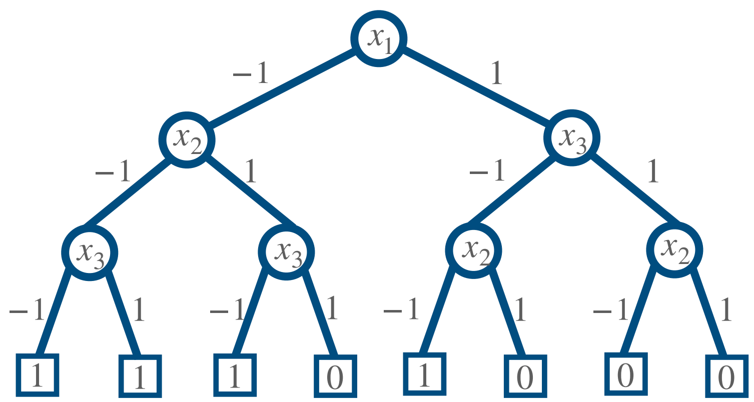
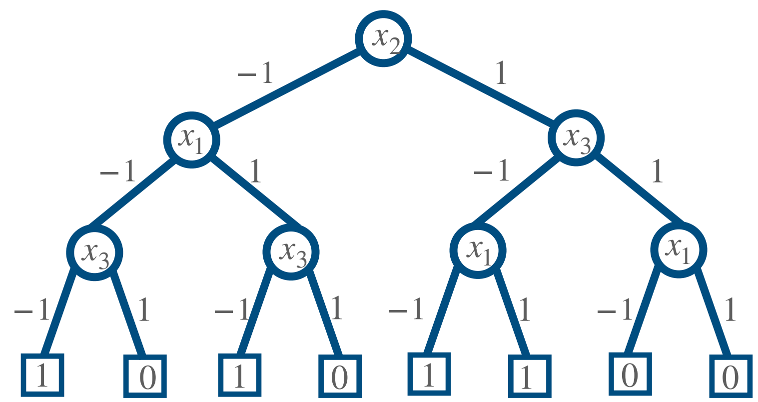


Randomized



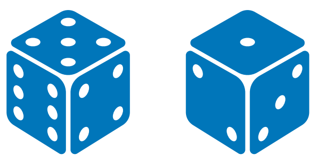
T_1



T_2

$\cdots$

Query complexity



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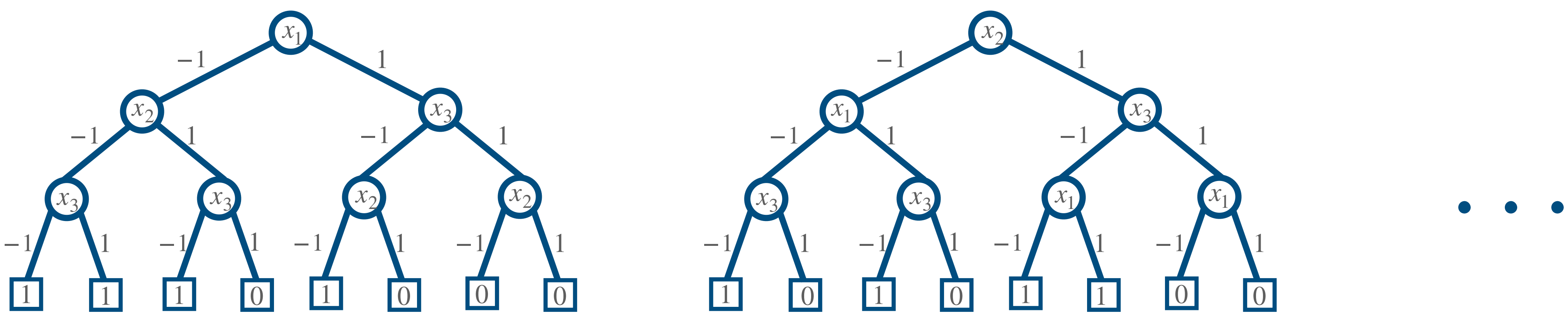
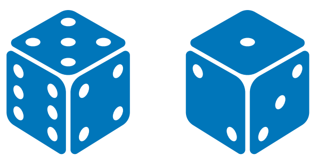


T_1 T_2

T computes $f: \{-1,1\}^n \to \{0,1\}$ with error ϵ if

$$\mathbf{P}_r[T_r(x) \neq f(x)] \leq \epsilon, \qquad \forall x \in \{-1,1\}^n.$$

Query complexity



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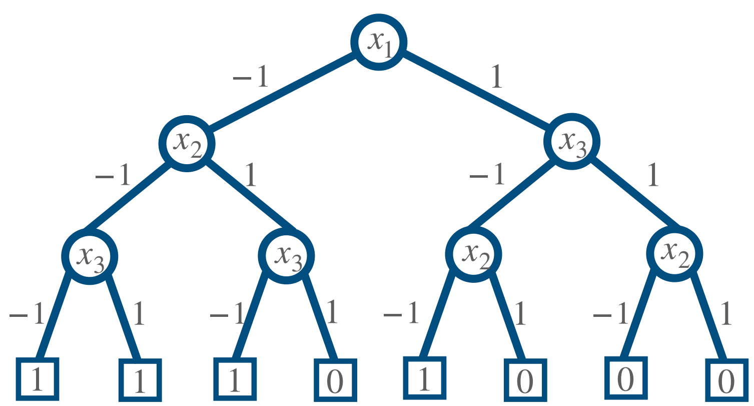

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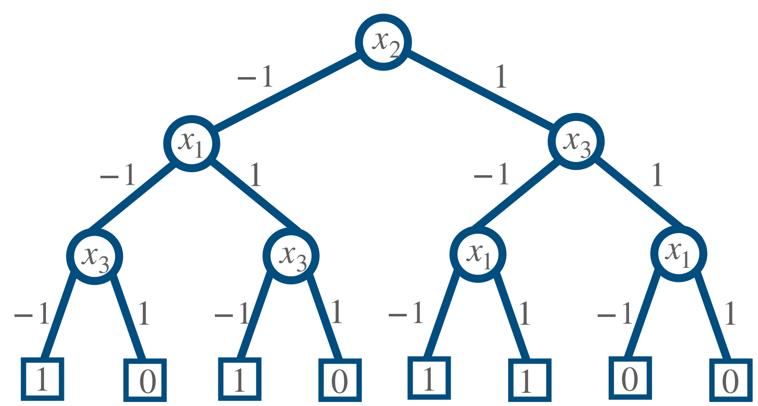

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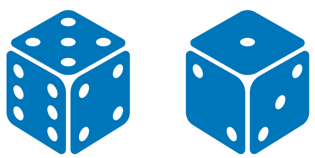

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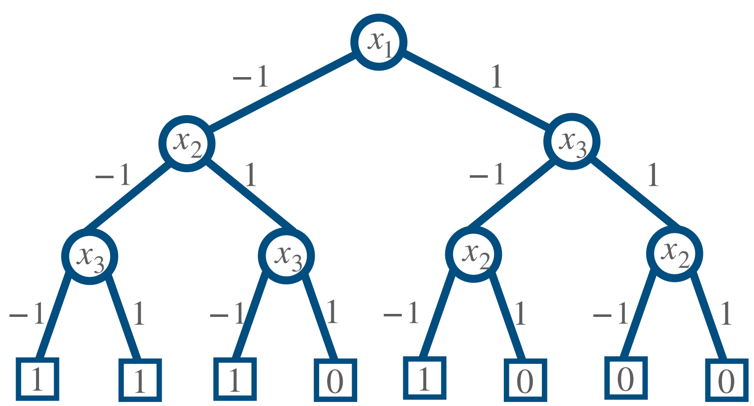

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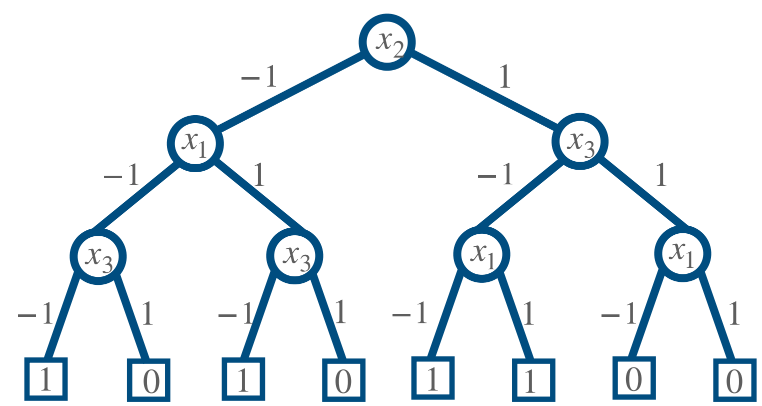

T_2

$\cdots$

$R_\epsilon(f) =$ minimum depth of a randomized decision tree for f with error ϵ.

Quantum query complexity

Quantum query

$$|\phi\rangle = \sum_{i,w} a_{i,w} |i\rangle |w\rangle$$

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$$\downarrow$$

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Quantum query complexity

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can access all x_i in a single query!

Quantum speedups

Query model captures nearly all quantum breakthroughs:

Deutsch-Jozsa's algorithm

Simon's algorithm

Grover's search

Bernstein-Vazirani's algorithm

Shor's factoring algorithm

……

Quantum speedups

Reference	Randomized	Quantum
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Largest possible separation?

[Buhrman et al. 02, Aaronson-Ambainis 15]

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Impossible!

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Optimal

Our results

Main Theorem.

Let k be any positive integer, $k \leq \dfrac{1}{3}\log n$. Then there is $f_k : \{-1,1\}^n \to \{0,1,*\}$ such that

$$Q_{\frac{1}{2}-\frac{1}{2^{k+4}}}(f_k) \leq \left\lceil \frac{k}{2} \right\rceil,$$

$$R_{\frac{1}{2^{k+1}}}(f_k) \geq \Omega\left(\frac{n^{1-\frac{1}{k}}}{(\log n)^{2-\frac{1}{k}}}\right).$$

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Corollary 1.

For any $\epsilon > 0$, there is $f : \{-1,1\}^n \to \{0,1,*\}$ with

$$Q_{1/3}(f) = O(1),$$

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Corollary 2.

For any monotone $\alpha : \mathbb{N} \to \mathbb{N}$, there is $f : \{-1,1\}^n \to \{0,1,*\}$ with

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Take $k = k(n)$ an arbitrarily slow-growing function, e.g. $k = \log\log\log n$.

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Our results: communication

Partial functions $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1,*\}$,

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Klartag-Regev 10	$R(f) = \tilde{\Omega}(n^{1/3})$	$O(\log n)$
Aaronson-Ambainis 15	$R(f) = \tilde{\Omega}(n^{1/2})$	$O(\log n)$
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lifting from query model

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Total functions $f: \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$,

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Tal 19	$R(f) \geq \Omega(Q(f)^{8/3-o(1)})$
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Our results: Fourier weight

Theorem

For any decision tree $g : \{-1,1\}^n \to \{0,1\}$ of depth d,

$$\sum_{\substack{S \subseteq \{1,2,\ldots,n\}: \\ |S| = \ell}} |\hat{g}(S)| \leq c^{\ell} \sqrt{\binom{d}{\ell}(1 + \log n)^{\ell-1}} \,.$$

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- Settles conjecture by Tal (2019)
- Previous bounds trivial already at $\ell \geq \sqrt{d}$

Independent work by Bansal & Sinha

Bansal–Sinha	**Our work**

Independent work by Bansal & Sinha

Bansal–Sinha

stochastic calculus

Our work

Fourier analysis

Independent work by Bansal & Sinha

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stochastic calculus

- advanced machinery

Our work

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Our work

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- optimal Fourier weight of decision trees

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existential

The problem: rorrelation

Rorrelation

$U \in \mathbb{R}^{n \times n}$, orthogonal matrix

$x_1, x_2, \ldots, x_k \in \{-1,1\}^n$

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$$\phi_{n,k,U}(x_1, x_2, \ldots, x_k) = \frac{1}{n} \mathbf{1}^T D_{x_1} U D_{x_2} U \cdots U D_{x_k} \mathbf{1}$$

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$$f_{n,k,U}(x_1, x_2, \ldots, x_k) = \begin{cases} 1 & \phi_{n,k,U} > 2^{-k}, \\ 0 & |\phi_{n,k,U}| \leq 2^{-k-1}, \\ * & \text{otherwise}\,. \end{cases}$$

Rorrelation: quantum algorithms

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Theorem (Aaronson-Ambainis, Tal).
There is a quantum algorithm using $\lceil k/2 \rceil$ queries that accepts x with probability

$$\frac{\phi_{n,k,U}(x) + 1}{2}.$$

Rorrelation: classical lower bound

—the "indistinguishability" argument

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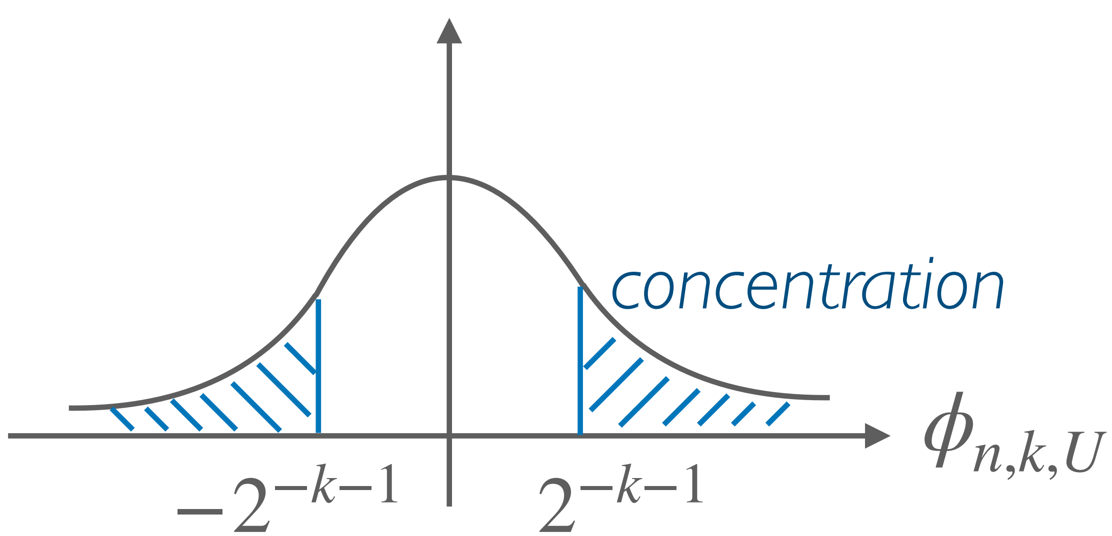

$$\mathbf{P}_{\mathscr{U}_{n,k}}[\phi > 2^{-k-1}] < 2^{-k-1}$$

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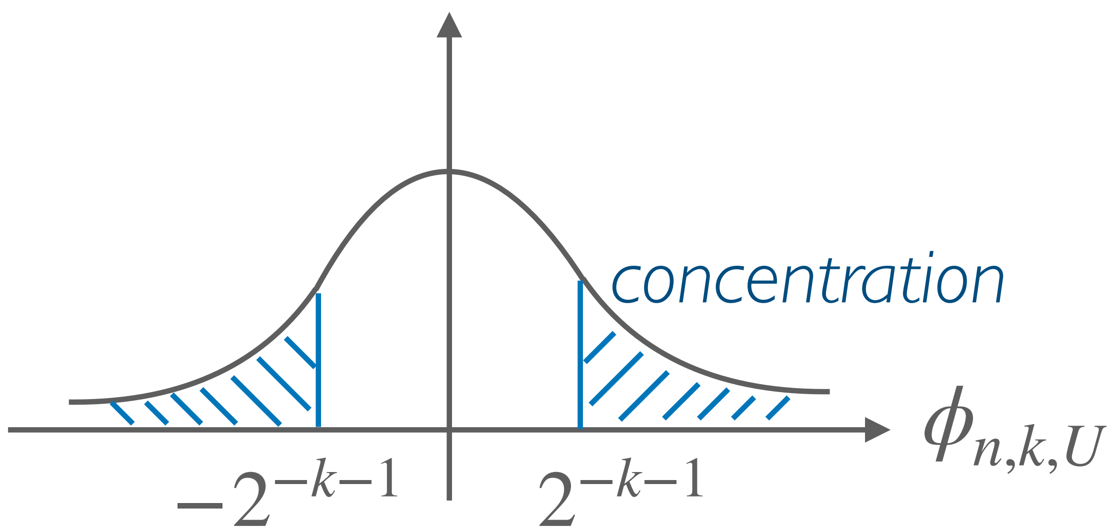

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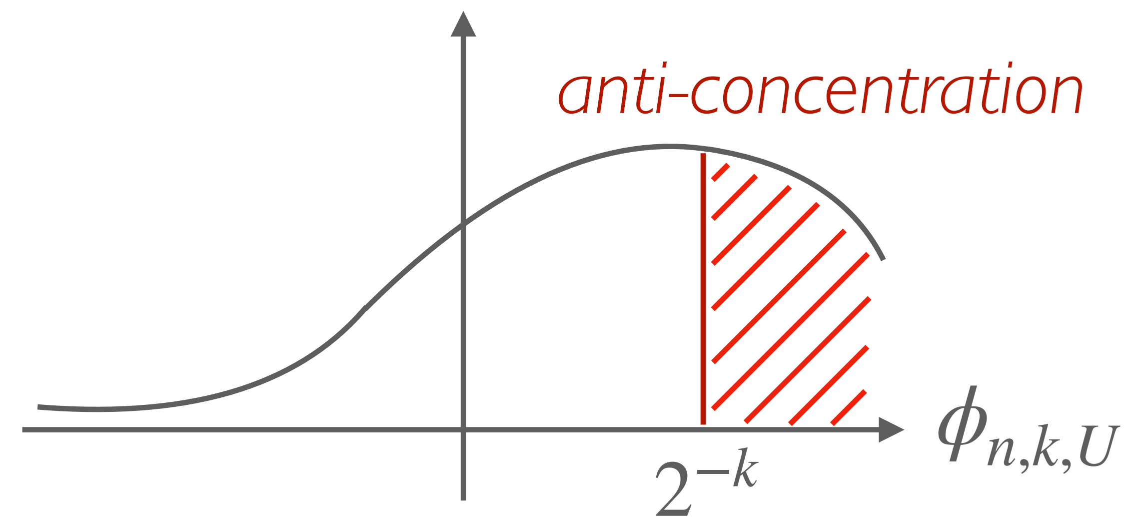

$\mathbf{P}_{\mathscr{D}_{n,k,U}}[\phi \geq 2^{-k}] \geq 2^{-k}$

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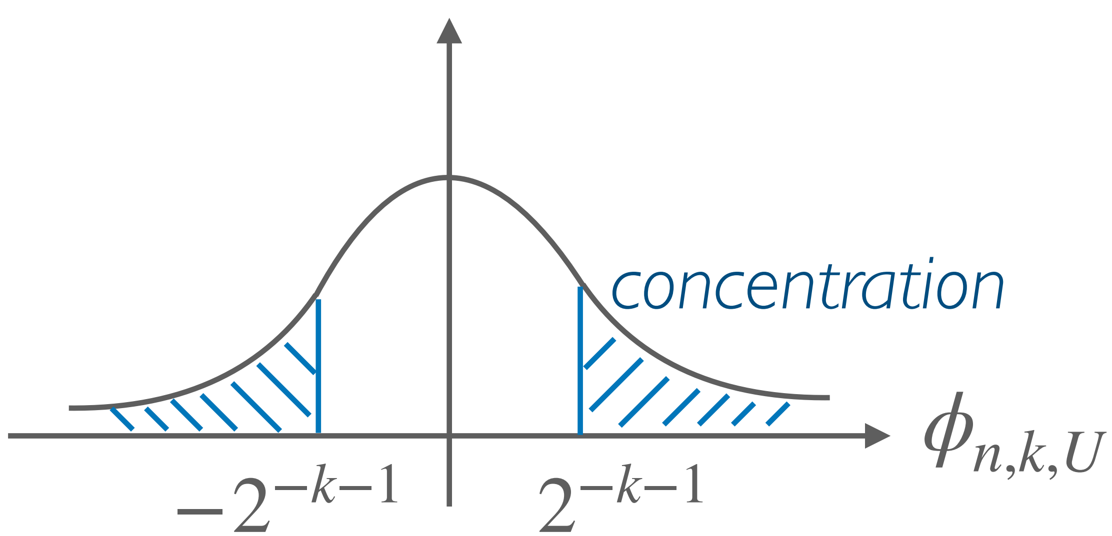

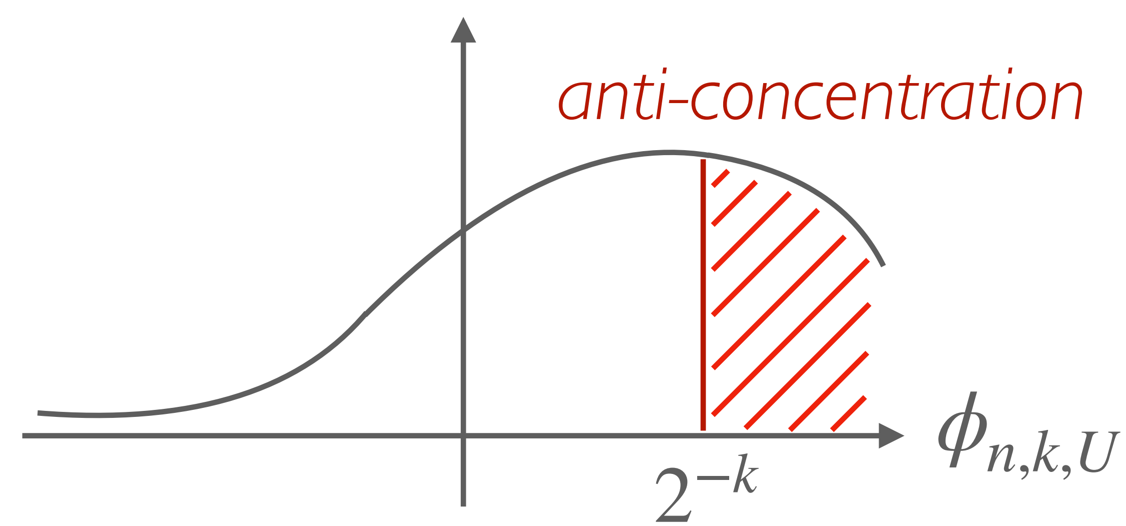

Thus, for any randomized query algorithm g of error ϵ,

$$\mathbf{E}_{\mathscr{D}_{n,k,U}} g(x) - \mathbf{E}_{\mathscr{U}_{n,k}} g(x) \geq 2^{-k-1} - 2\epsilon\,.$$

Rorrelation: classical lower bound

—the "indistinguishability" argument

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Rorrelation: classical lower bound

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$$\mathbf{E}_{\mathscr{D}_{n,k,U}} g(x) - \mathbf{E}_{\mathscr{U}_{n,k}} g(x)$$

$$\leq \left| \mathbf{E}_{\mathscr{D}_{n,k,U}} \sum_S \hat{g}(S)\chi_S - \mathbf{E}_{\mathscr{U}_{n,k}} \sum_S \hat{g}(S)\chi_S \right|$$

Rorrelation: classical lower bound

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$\ell = k$

[Tal]

$$\leq O\left(\frac{\ell \log n}{n}\right)^{\frac{\ell}{2}\frac{k-1}{k}}$$

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We prove: $\leq c^{\ell}\sqrt{\binom{d}{\ell}(\ln en)^{\ell-1}}$

Therefore,

$$R_{2\text{-}o(k)}(f_k) = \tilde{\Omega}(n^{1-\frac{1}{k}})\,. \quad \blacksquare$$

Fourier weight of decision trees

Fourier weight of decision trees

Theorem (this work).

For any decision tree $T : \{-1,1\}^n \to \{0,1\}$ of depth d,

$$\sum_{\substack{S \subseteq \{1,2,\ldots,n\}: \\ |S| = \ell}} |\hat{T}(S)| \le c^{\ell} \sqrt{\binom{d}{\ell}(1 + \log n)^{\ell - 1}}\,.$$

Fourier weight of decision trees

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$\|\!| L_\ell T |\!\|$

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Fourier weight of decision trees

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Fix any decision tree $T : \{-1,1\}^n \to \{-1,0,1\}$ of depth d, and $\mathbf{P}[T(x) \neq 0] = p$. Then

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Our approach

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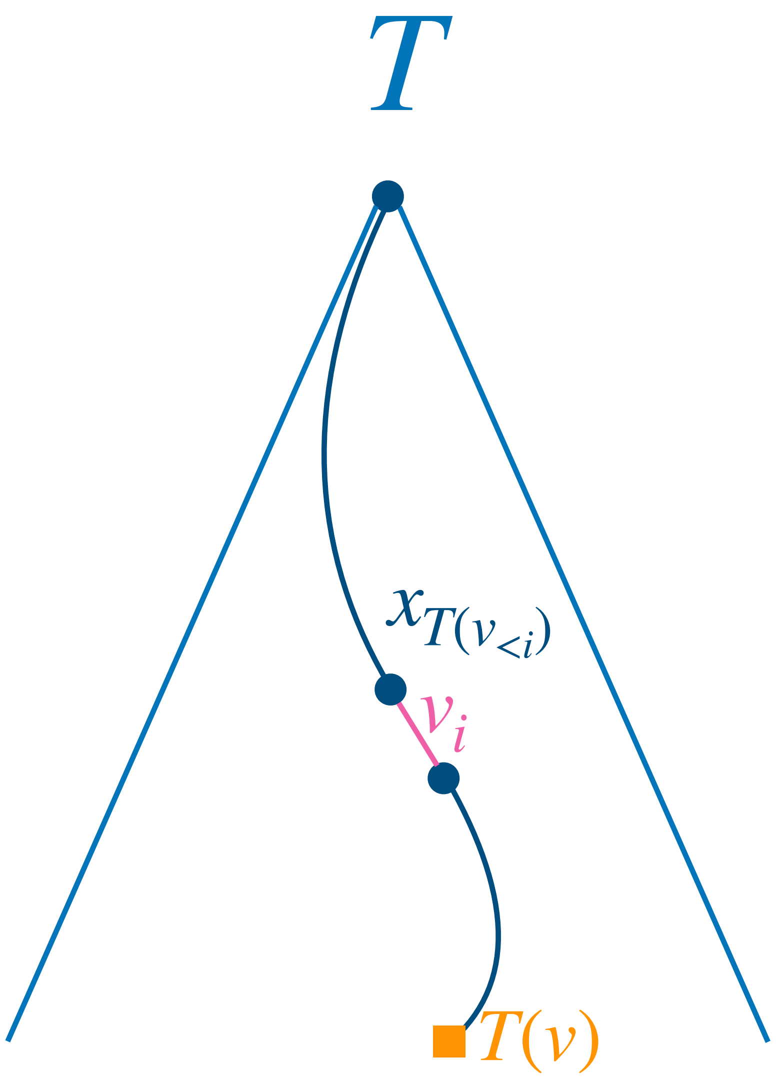
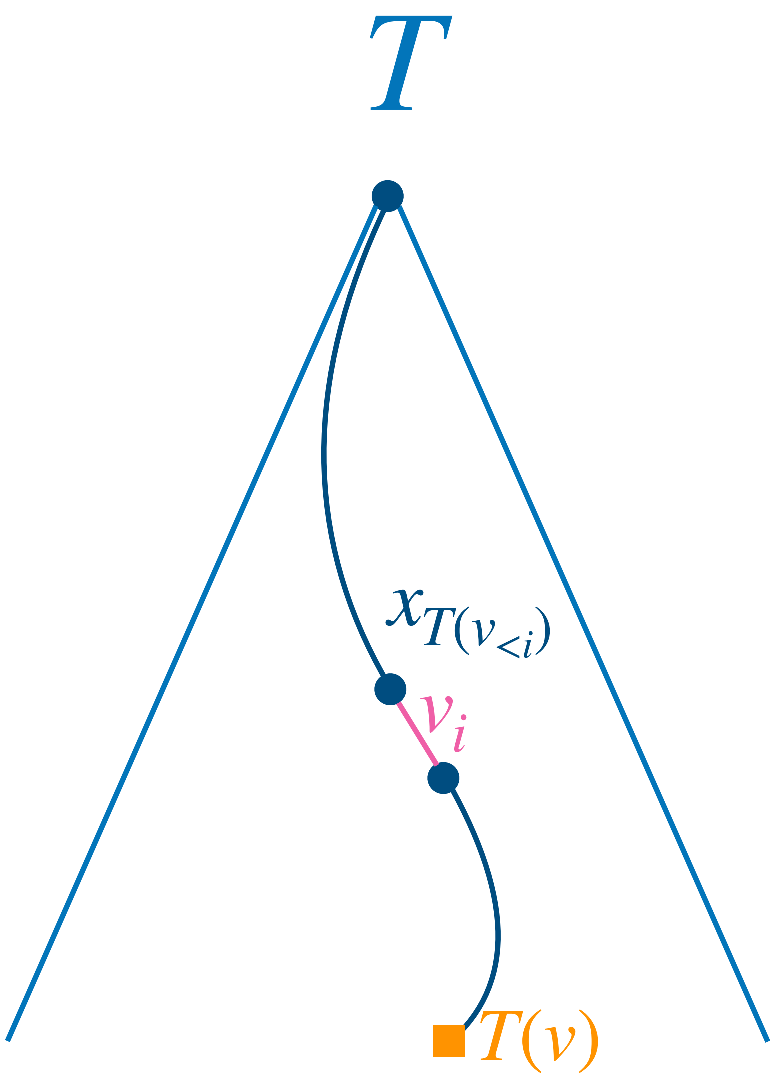


$v \in \{-1,1\}^d$

Our approach



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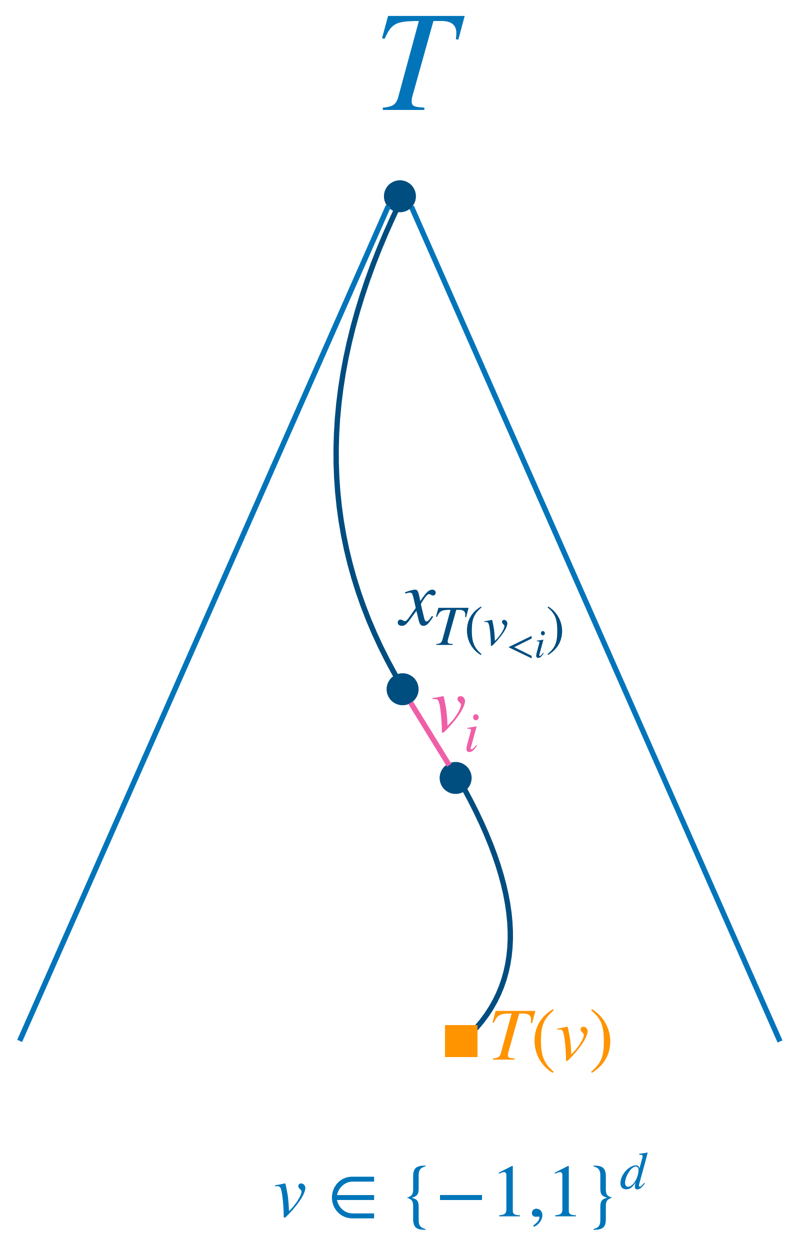

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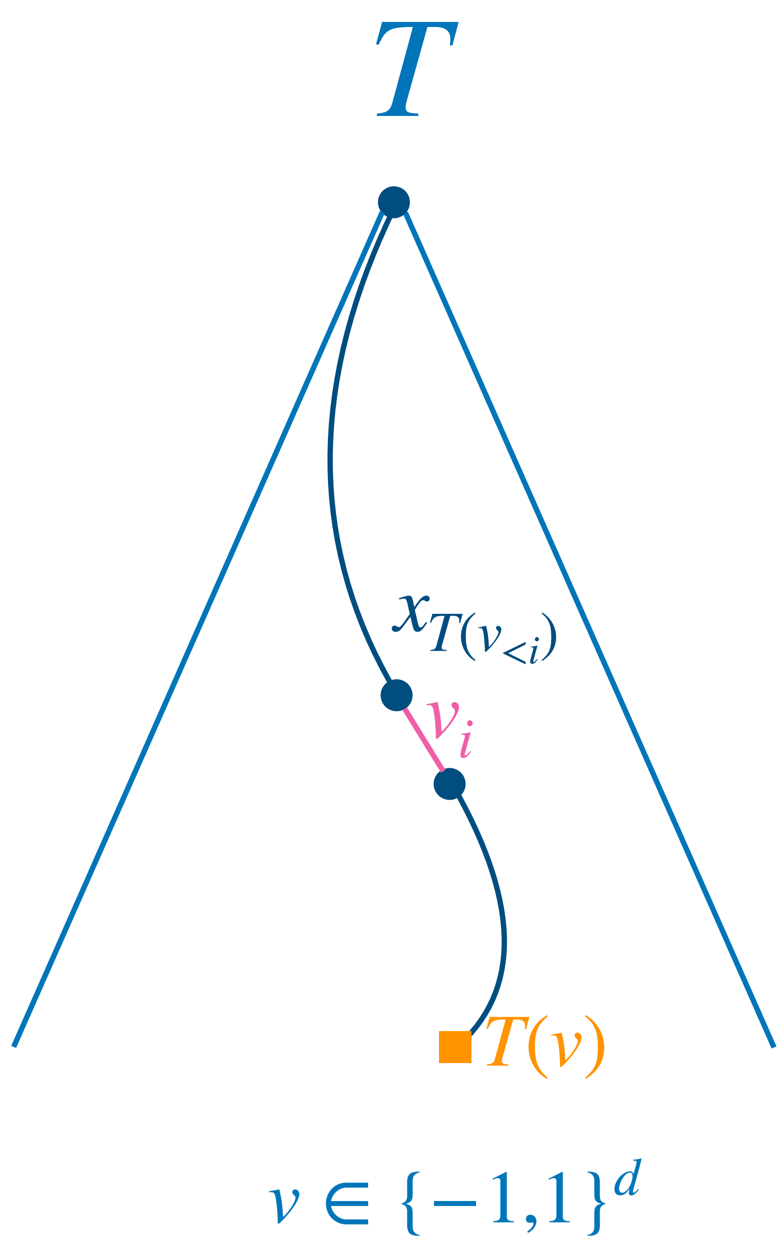

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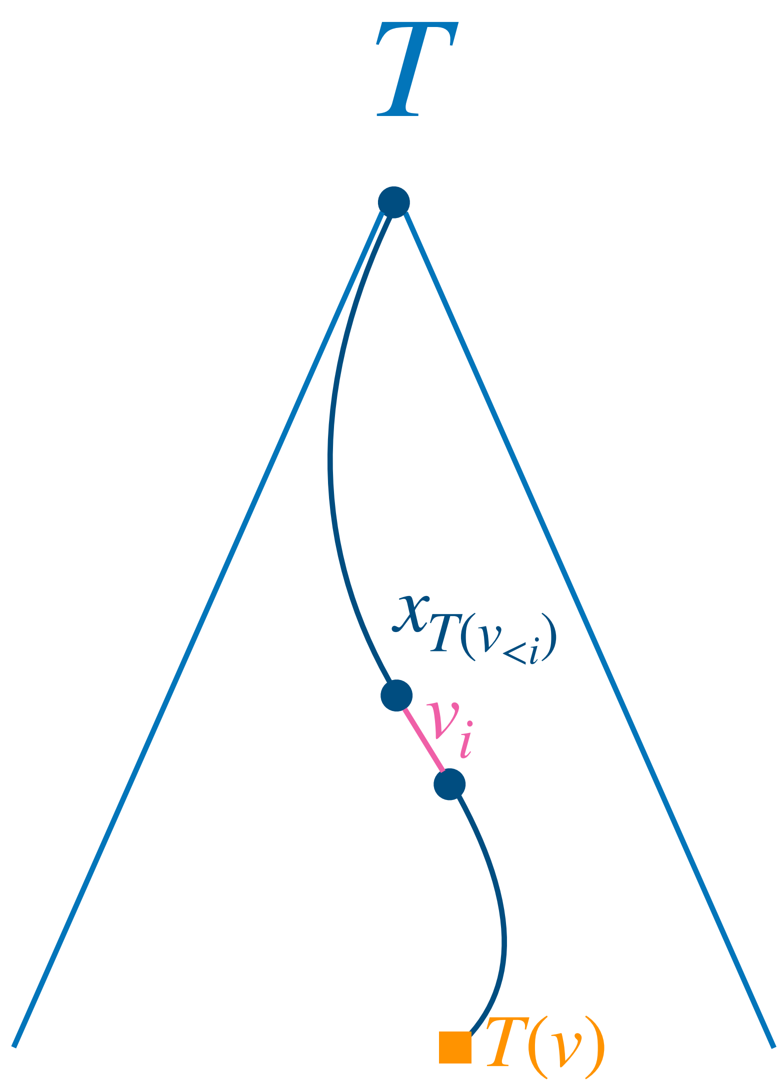

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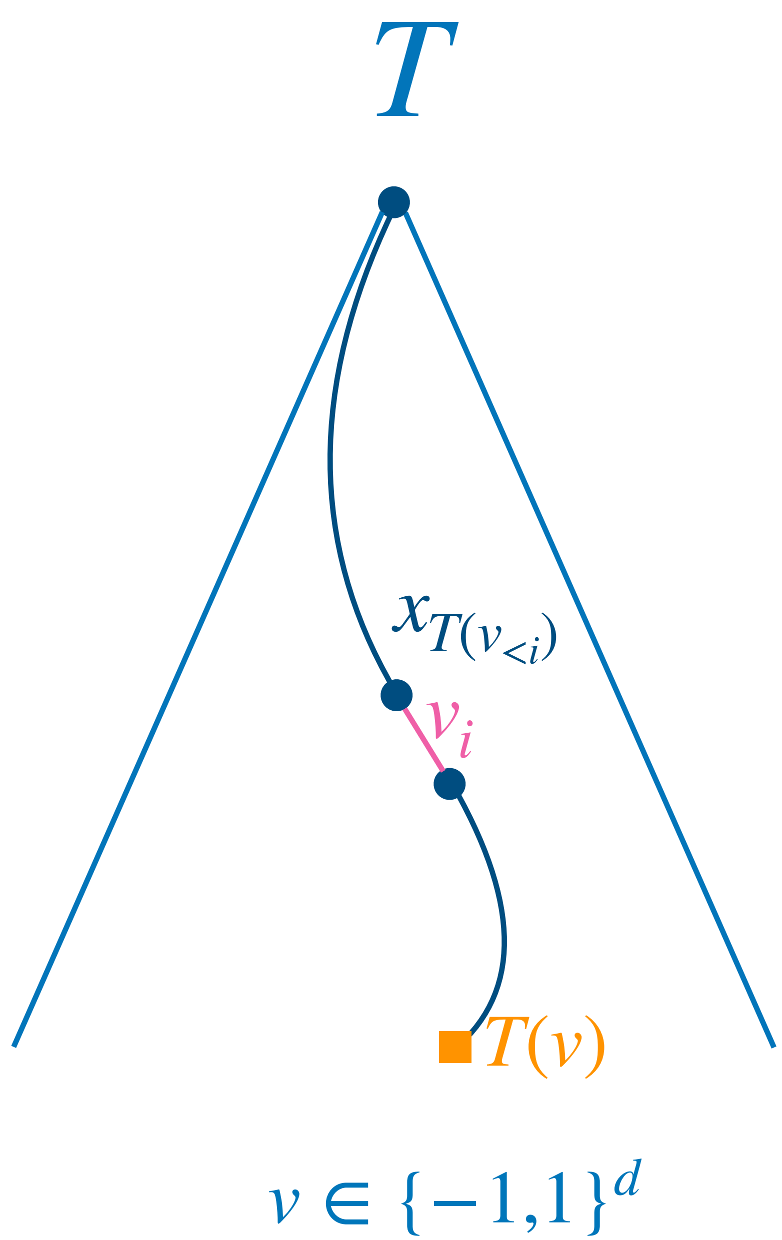

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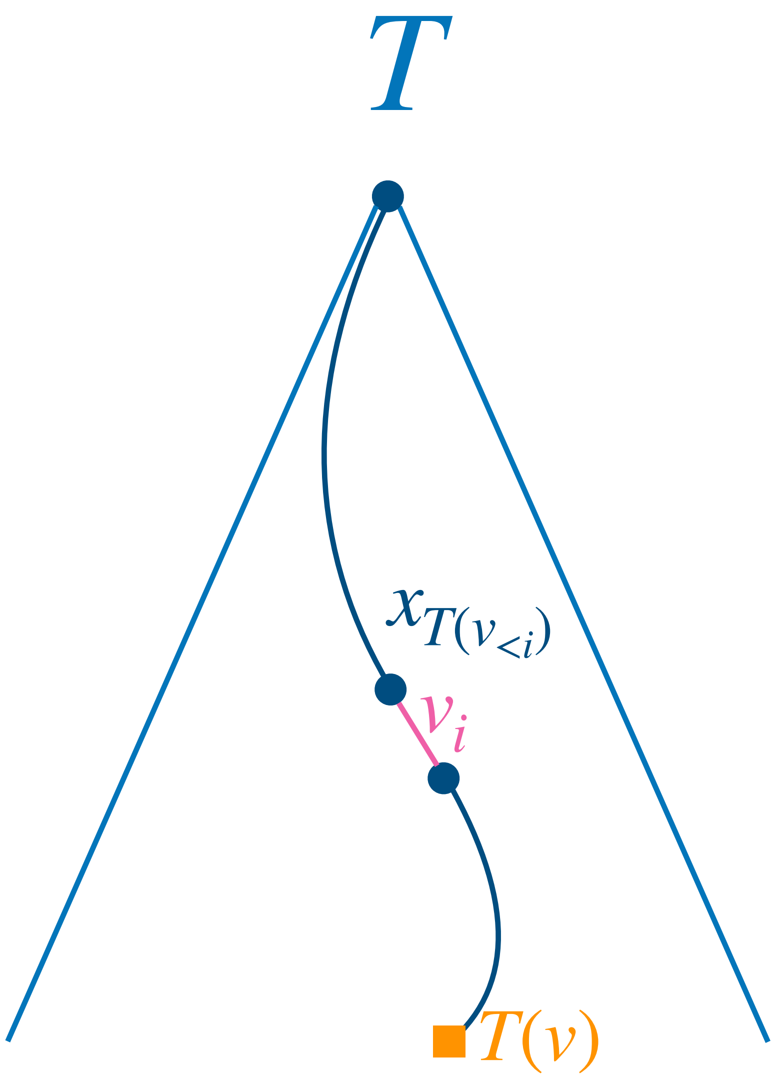

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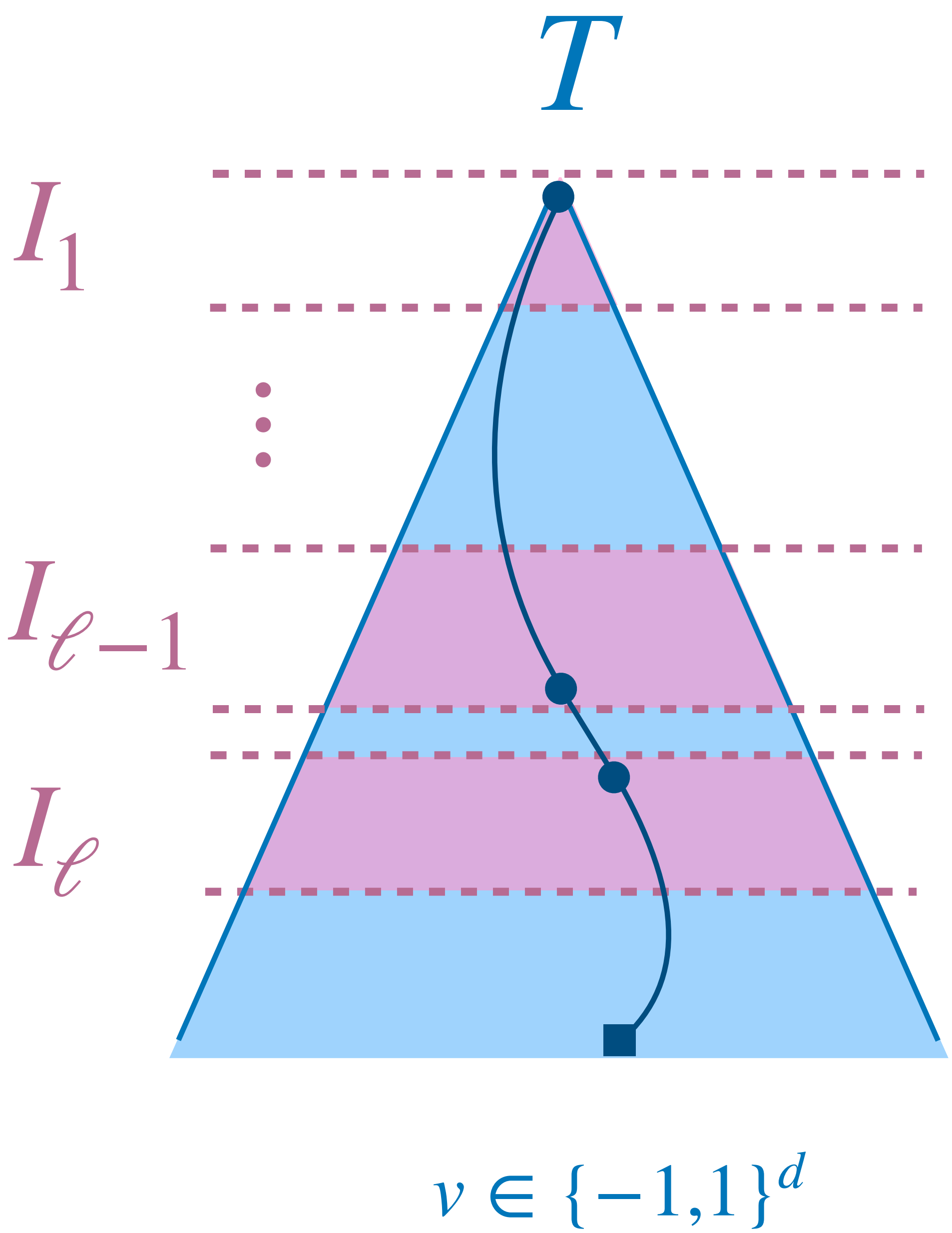

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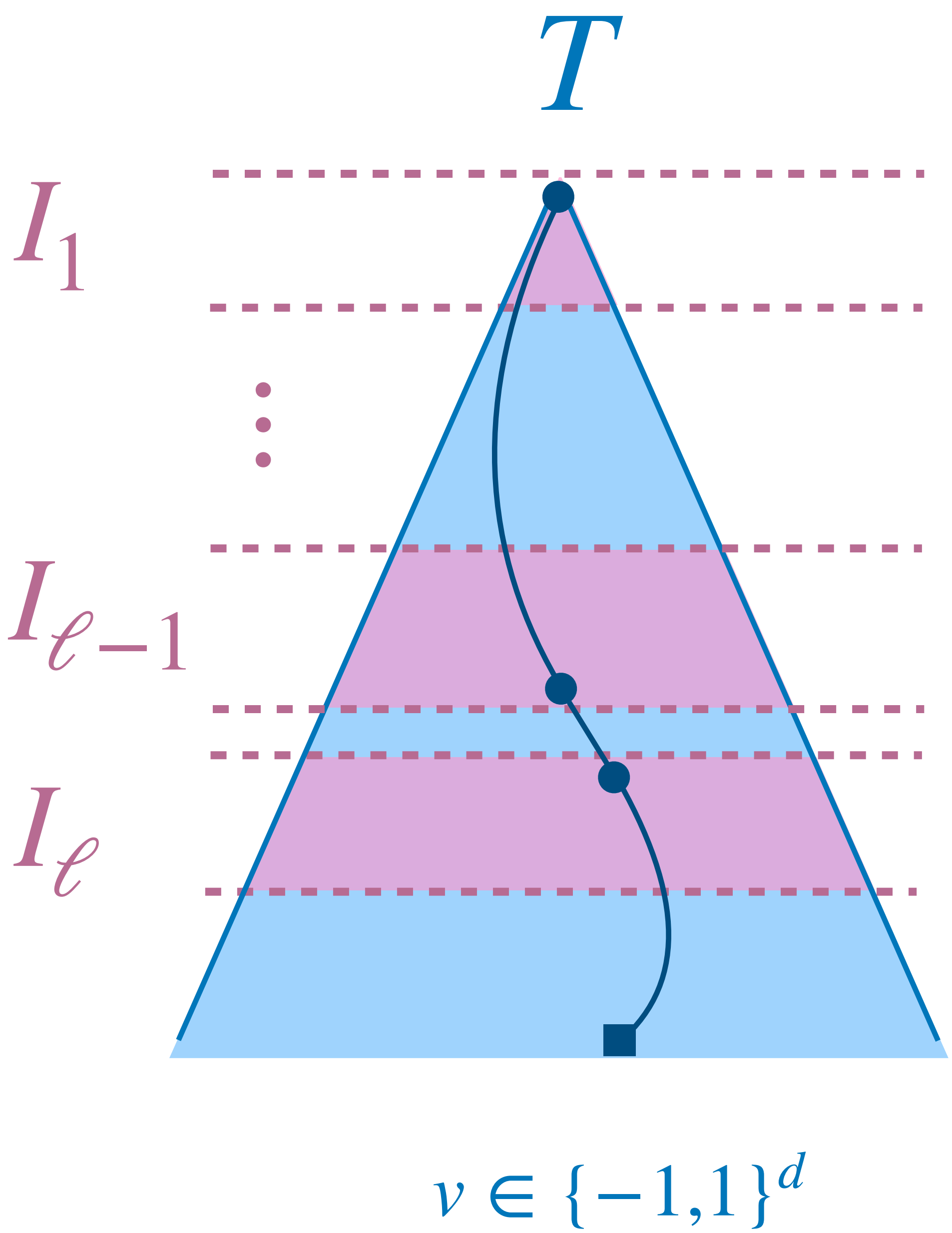

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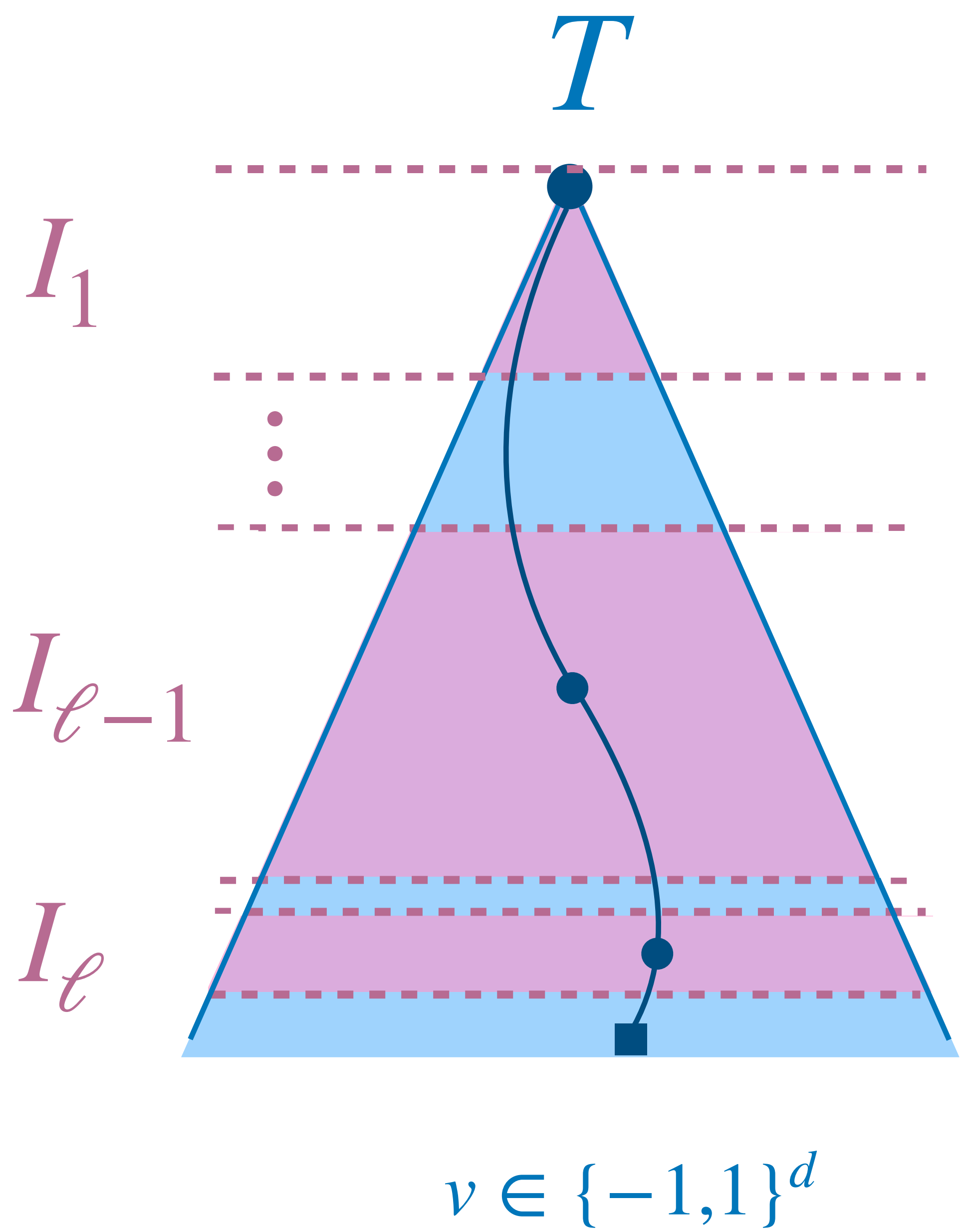

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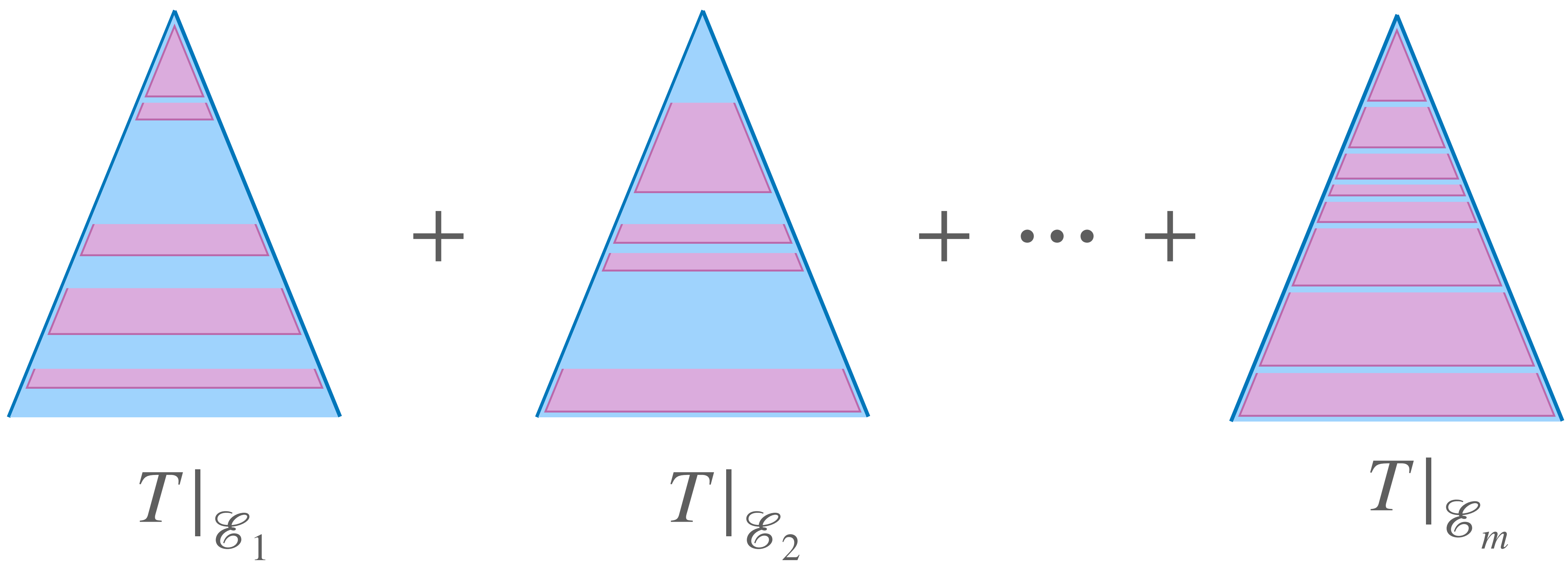
Our key idea

$L_\ell T =$



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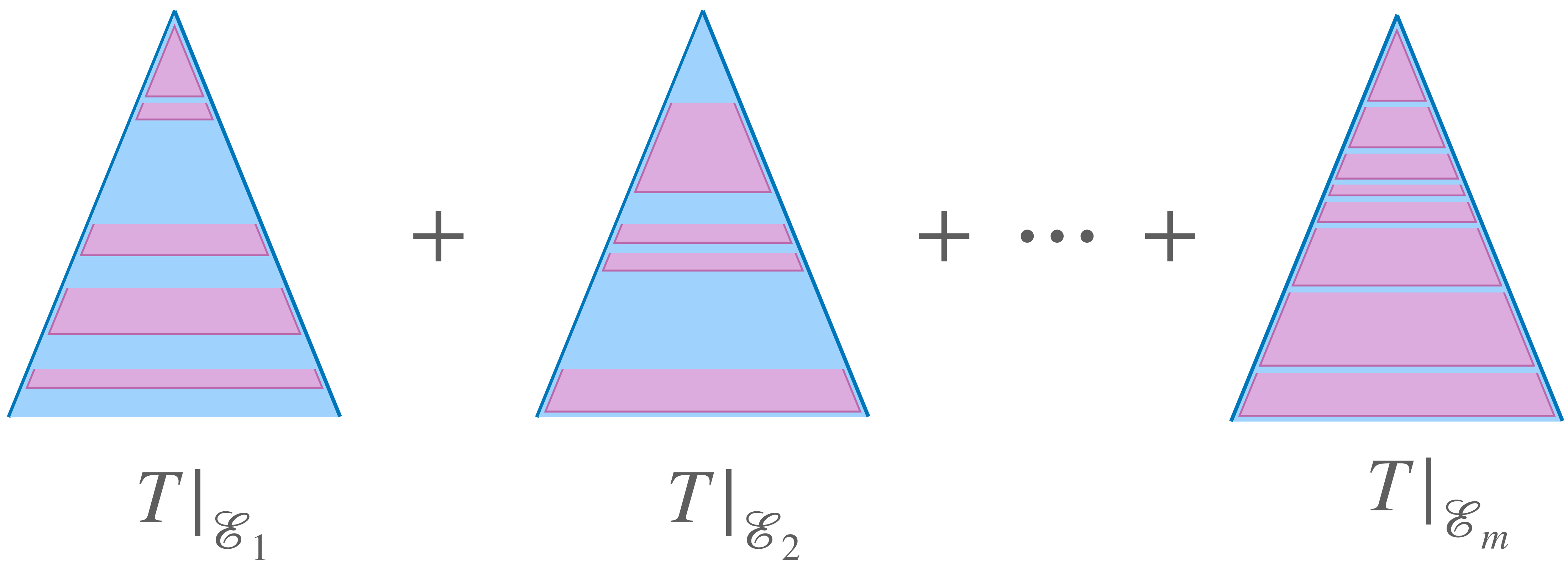

$$\||L_\ell T\|| \leq \sum \||T|_{\mathscr{E}_i}\|| .$$ (Triangle-inequality)

Our proof

$$|||L_\ell T||| \leq \sum_i |||T|_{\mathscr{E}_i}||| \,.$$

Our proof

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Theorem 1.

For some absolute constant c, and any elementary family $\mathscr{E} = I_1 * I_2 * \cdots * I_\ell$,

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$\mathscr{P}_{d,\ell}$ can be partitioned into elementary families $\mathscr{E}_1, \mathscr{E}_2, \ldots, \mathscr{E}_m$ s.t. for some const C,

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Thank you!